Proceedings of Symposia in Pure Mathematics
Volume 54 (1993), Part 2

Einstein Metrics on Circle Bundles

GARY R. JENSEN AND MARCO RIGOLI

ABSTRACT. We give a new method for constructing Riemannian metrics on fiber bundles. On the tangent bundle we obtain the Sasaki metric. If the structure group does not act transitively on the fiber, then the constructed metric depends on arbitrary functions on the orbit space. The Einstein condition on the constructed metric imposes differential equations with appropriate boundary values on these arbitrary functions. In some cases these equations can be solved. We show how it works for circle bundles over the two dimensional sphere.

Introduction

In 1978 T. Eguchi and A. J. Hanson [4] constructed a complete Einstein metric on $T^*\mathbf{C}P^1$. In the same year D. Page [7] constructed an Einstein metric on the nontrivial S^2 bundle over S^2. In 1979 E. Calabi [3] found Einstein metrics on $T^*\mathbf{C}P^n$ for $n \geq 1$. In 1981 L. Berard-Bergery [1] and [2] constructed Einstein metrics on complex line bundles and S^2-bundles over Einstein Kähler manifolds in the context of a general construction on cohomogeneity one spaces. In 1987 D. Page and C. N. Pope [8] produced the same constructions from quite a different point of view.

In this paper we present a metric construction on fiber bundles. In the case where the structure group of the bundle is the circle this construction gives a new method for obtaining the above Einstein metrics. As the method works in much more generality, it may lead to the construction of compact Einstein spaces of higher cohomogeneity.

1. The metric construction

The constuction begins with the following data:

(1) M, ds^2 is a Riemannian manifold
(2) G is a Lie group with Lie algebra $\mathscr{G}$

1991 *Mathematics Subject Classification.* Primary 53C25.
Key words and phrases. Einstein metrics, fiber bundles.
This paper is in final form and no version of it will be submitted for publication later.

(3) $\pi : P \to M$ is a principal G-bundle
(4) ω is a $\mathcal{G}$-valued connection form on P
(5) F is a manifold on which G acts
(6) g is a G-invariant Riemannian metric on F
(7) h is a positive G-invariant C^∞ function on F.

The action of G on F, g induces a homomorphism $\hat{}$ of $\mathcal{G}$ into the Lie algebra of Killing vector fields $\mathcal{J}(F, g)$, given by: if $X \in \mathcal{G}$ and $x \in F$, then $\hat{X}(x) = \frac{d}{dt}|_0 e^{tX}x$. If we let $\hat{\omega} = \hat{} \circ \omega : TP \to \mathcal{J}(F, g)$, then for any 1-form φ on F it follows that $\varphi(\hat{\omega})$ is a 1-form on P.

If $\{\varphi^a\}$ is a local orthonormal coframe field in F, g, then we define a local symmetric quadratic form q on $P \times F$ by

$$q = h^2 \pi^* ds^2 + \sum_a (\varphi^a + \varphi^a(\hat{\omega}))^2.$$

As this form does not depend on the choice of orthonormal coframe field $\{\varphi^a\}$, it is globally defined on $P \times F$.

The group G acts on $P \times F$ by $a(p, x) = (pa^{-1}, ax)$. The quotient, denoted $PF = P \times_B F$, is the fiber bundle associated to P with standard fiber F. In **[5]** we proved the following result.

THEOREM. *The form q is G-invariant, horizontal and $q(v, v) = 0$ if and only if v is a vertical tangent vector of $P \times F$. Hence it descends to a Riemannian metric ds^2_{PF} on PF such that $\sigma^* ds^2_{PF} = q$, where $\sigma : P \times F \to PF$ is the projection map.*

REMARKS. 1. In our approach here the curvature computations take place on $P \times F$ which is the natural home of all the data. The major problem is to describe the G-invariant metrics and functions on F. This, of course, involves functions on the orbit space $G\backslash F$.

2. This construction, with $h = 1$ and g the standard metric on $\mathbf{R}^n$, produces the Sasaki metric on the tangent bundle TM.

3. In 1979 J. Nash **[6]** studied metrics defined on $P \times_G F$ so that σ is a Riemannian submersion. The metric he used on $P \times F$ is

$$\pi^* ds^2 + < \omega, \omega > + g,$$

where $<, >$ is a biinvariant metric on G. If G acts transitively on F and $F, g = G/H$ is a normal homogeneous space, then his metric and ours are the same. In general, say when G does not act transitively on F, the two metrics are different.

2. Bundles over the two-sphere

In order to simplify the exposition for this short note we will carry out the above construction for the case of the Hopf fibration $\pi : S^3 \to S^2$. As long as the structure group of the bundle is S^1 the details of the computations are not changed significantly.

On S^2 we take the standard metric ds^2 of constant curvature equal to four.

The Hopf fibration is a principal S^1-bundle, where $S^1 \subset \mathbf{C}$ as the complex numbers of modulus one, and where $S^3 \subset \mathbf{C}^2$ as the set of unit vectors with respect to the standard hermitian inner product $<,>$ on $\mathbf{C}^2$. Here S^1 acts on the right on S^3 by scalar multiplication from the right on $\mathbf{C}^2$. For the connection form on this bundle we take

$$\omega_{(p)}X = \langle p, X \rangle = \bar{p}^t X,$$

which takes values in $i\mathbf{R}$, the Lie algebra of S^1. It is convenient to set $\omega = iA$, so that A is a real valued 1-form on S^3.

On S^3 there is a complex valued 1-form defined at $p = \binom{a}{b} \in S^3$ on $X = \binom{u}{v} \in T_pS^3$ by

$$(\theta^1 + i\theta^2)_p X = av - bu.$$

It has the property that

$$\pi^* ds^2 = (\theta^1)^2 + (\theta^2)^2.$$

It is easily verified that

$$dA = 2\theta^1 \wedge \theta^2.$$

For each integer k we let $\rho_k : S^1 \to S^1$ be the representation $\rho_k(a) = a^k$.

Let F be either $\mathbf{C}$ or $S^2 \subset \mathbf{C} \times \mathbf{R}$. The action of S^1 on F is by multiplication of $S^1 \subset \mathbf{C}$ on $\mathbf{C}$. Then

$$E_k = S^3 \times_{\rho_k} \mathbf{C}$$

is a complex line bundle over S^2 with Chern number equal to $-k$; and

$$S^3 \times_{\rho_1} S^2$$

is the nontrivial S^2 bundle over S^2.

Using polar coordinates, we can write any complete G-invariant metric on F as

$$g = dr^2 + f(r)^2 d\theta^2.$$

An arbitrary G-invariant function on F is given by $h(r)$. Of course both $f(r)$ and $h(r)$ must be positive in the interior of their domains, I_F. The boundary conditions which they must satisfy depend on the two cases as follows.

In case $F = \mathbf{C}$, then $I_F = 0 \le r < \infty$ and

(1) h is an even function at $r = 0$ in the sense that its odd order derivatives are zero there;

(2) f is an odd function at $r = 0$ in the sense that its even order derivatives are zero there;

(3) $\dot{f}(0) = 1$.

In case $F = S^2$, then $I_F = 0 \leq r \leq \pi$ and

(1) h is an even function at $r = 0$ and at $r = \pi$;
(2) f is an odd function at $r = 0$ and at $r = \pi$;
(3) $\dot{f}(0) = 1$ and $\dot{f}(\pi) = -1$.

Our symmetric bilinear form on $S^3 \times F$ becomes

$$q = h(r)^2 \pi^* ds^2 + dr^2 + f(r)^2 (d\theta + kA)^2.$$

By our theorem q descends to a complete metric ds^2_{PF} on E_k or $S^3 \times_{\rho_1} S^2$, as the case may be.

REMARK. It turns out to be sufficient to consider the form

$$\tilde{q} = h(r)^2 \pi^* ds^2 + dr^2 + k^2 f(r)^2 A^2$$

on $S^3 \times I_F$. This reduction is a crucial step in making the calculations on more complicated examples, say when the structure group is nonabelian.

3. The Einstein equations

The 1-forms on $S^3 \times F$

$$\psi^1 = h\theta^1, \qquad \psi^2 = h\theta^2, \qquad \psi^3 = dr, \qquad \psi^4 = f(d\theta + kA)$$

diagonalize q. If they are pulled back to PF by a local section of σ, they form a local orthonormal coframe field for ds^2_{PF}.

Using this coframe, we find the components of the Ricci tensor of ds^2_{PF} to be:

$$R_{ab} = 0, \quad \text{if} \quad a \neq b,$$

$$R_{11} = R_{22} = \frac{4}{h^2} - \left(\frac{\dot{h}}{h}\right)^2 - 2\left(\frac{kf}{h^2}\right)^2 - \frac{\ddot{h}}{h} - \frac{\dot{f}\dot{h}}{fh}$$

$$R_{33} = -2\frac{\ddot{h}}{h} - \frac{\ddot{f}}{f}$$

$$R_{44} = 2\left(\frac{kf}{h^2}\right)^2 - 2\frac{\dot{f}\dot{h}}{fh}.$$

Then ds^2_{PF} is Einstein if and only if

$$\frac{4}{h^2} - \left(\frac{\dot{h}}{h}\right)^2 - 2\left(\frac{kf}{h^2}\right)^2 + \frac{\ddot{h}}{h} + \frac{\ddot{f}}{f} - \frac{\dot{f}\dot{h}}{fh} = 0$$

$$\frac{\ddot{h}}{h} + \left(\frac{kf}{h^2}\right)^2 - \frac{\dot{f}\dot{h}}{fh} = 0.$$

We refer to these as the Einstein equations. The Einstein constant will be

$$\lambda = -2\frac{\ddot{h}}{h} - \frac{\ddot{f}}{f}.$$

Multiplying the second of the Einstein equations by $\frac{h}{f}$, rearranging and then multiplying by $2\frac{\dot{h}}{f}$, we find that

$$\left[\left(\frac{\dot{h}}{f}\right)^2 - \frac{k^2}{h^2}\right]^{\cdot} = 0.$$

Hence we have a first integral which we write as

$$\left(\frac{\dot{h}}{f}\right)^2 - \frac{k^2}{h^2} = k^2 a,$$

where a is the constant of integration. Thus any solution of the Einstein equations must satisfy

$$f = \frac{|\dot{h}|h}{|k|\sqrt{1 + ah^2}}.$$

The solutions of these equations with each set of boundary conditions are derived in detail in §11 of **[1]**. Complete solutions restrict the allowable values of k.

4. Ricci flat, complete, Kähler metrics on $E_{\pm 2}$

Throughout this section we set $\varepsilon = \pm 1$. On $S^3 \times \mathbf{C}$ we set

$$\varphi^1 = \psi^1 + i\psi^2$$
$$\varphi^2 = \psi^3 + i\varepsilon\psi^4.$$

It is easy to see that these descend to a pair of integrable almost complex structures on PF which we shall refer to as J_ε. The Kähler form of J_ε is (the pull back by any section of σ of)

$$\kappa = h^2\theta^1 \wedge \theta^2 + \varepsilon f\, dr \wedge (d\theta + kA).$$

Thus

$$d\kappa = 2(h\dot{h} - \varepsilon k f) dr \wedge \theta^1 \wedge \theta^2,$$

from which we see that ds^2_{PF}, J_ε is Kähler if and only if

$$h\dot{h} = \varepsilon k f. \tag{1}$$

Theorem. *Let k be a nonzero integer and let $\varepsilon = \operatorname{sign}(k)$ so that $\varepsilon k > 0$. Then the metric $ds^2_{E_k}$ on the complex line bundle E_k is Ricci flat, complete, and Kähler if and only if*

$$h\ddot{h} + 2\dot{h}^2 = 2 \tag{2}$$
$$\varepsilon k f = h\dot{h}, \tag{3}$$

where f and h are positive on $0 < r < \infty$ and satisfy the appropriate initial conditions described above. In particular, it follows that $\varepsilon k = 2$. This problem, with the normalization $h(0) = 1$, has a unique solution on $0 \le r < \infty$.

PROOF. Combining the Kähler condition (1) with the Einstein equations, we have

$$h\dot{h} = \varepsilon k f \tag{4}$$

$$2\frac{\ddot{h}}{h} + \frac{\ddot{f}}{f} = 0 \tag{5}$$

$$2\left(\frac{kf}{h^2}\right)^2 - 2\frac{\dot{f}\dot{h}}{fh} - \frac{\ddot{f}}{f} = 0 \tag{6}$$

$$\frac{4}{h^2} - \left(\frac{\dot{h}}{h}\right)^2 - 2\left(\frac{kf}{h^2}\right) - \frac{\ddot{h}}{h} - \frac{\dot{f}\dot{h}}{fh} = 0. \tag{7}$$

One can show, under our positivity and initial conditions, that functions f and h are solutions of equations (4)–(7) if and only if they are solutions of (2) and (3).

From (2) and the fact that $\dot{h}(0) = 0$, it follows that $h(0)\ddot{h}(0) = 2$. Substituting this into the derivative of (3) and using the condition that $\dot{f}(0) = 1$, it follows that $\varepsilon k = 2$.

Finally, it is an elementary exercise to show that the initial value problem $h\ddot{h} + 2\dot{h}^2 = 2$, $h(0) = 1$ and $\dot{h}(0) = 0$ has a unique solution. It is defined and strictly increasing on $0 \leq r < \infty$, and it is an even function at 0.

It follows then that $ds^2_{E_k}$ is complete since h and f are both positive on $0 < r < \infty$ and thus g on $\mathbf{C}$ is complete. □

REMARK. The bundle E_2 is the cotangent bundle, T^*S^2. The above Ricci flat, Kähler metric on E_2 was found in [**4**] and [**1**]. Our metric on $E_{-2} = TS^2$ seems to be new.

REFERENCES

1. L. Berard-Bergery, *Sur de nouvelles variétés riemanniennes d'Einstein*, Publications de l'Institut E. Cartan **4** (1982), 1–60, Nancy.
2. A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.
3. E. Calabi, *Métriques kähleriennes et fibrés holomorphes*, Ann. Ecol. Norm. Sup. **12** (1979), 269–294.
4. T. Eguchi and A. J. Hanson, *Asymptotically flat self-dual solutions to Euclidean Gravity*, Phys. Lett. **237** (1978), 249–251.
5. G. R. Jensen and M. Rigoli, *Harmonic Gauss maps*, Pac. J. of Math. **136** (1989), 261–282.
6. J. C. Nash, *Positive Ricci curvature on fibre bundles*, J. Diff. Geom. **14** (1979), 241–254.
7. D. Page, *A compact rotating gravitational instanton*, Phys. Lett. **79 B** (1979), 235–238.
8. D. Page and C. Pope, *Inhomogeneous Einstein metrics on complex line bundles*, Classical Quantum Gravity **4** (1987), 213–225.

WASHINGTON UNIVERSITY

UNIVERSITÀ DI MILANO, ITALY