Vol. 42, No. 3 DUKE MATHEMATICAL JOURNAL© September 1975

IMBEDDINGS OF STIEFEL MANIFOLDS INTO GRASSMANNIANS

GARY R. JENSEN

Introduction. A. Weinstein observed in (3) that certain metrics constructed on odd-dimensional spheres by M. Berger and I. Chavel as counterexamples to an extension of a lemma of Klingenberg arise very naturally as the induced metric on distance spheres in complex projective space with the Fubini-Study metric. From Weinstein's observation it is natural to expect that some of the spaces and metrics of [1], namely the metrics on Stiefel manifolds, arise in an analogous fashion from imbeddings of the Stiefel manifolds into Grassmannians, the latter with the canonical Riemannian metric.

In this paper we study a class of equivariant imbeddings of Stiefel manifolds into Grassmannians. The imbedded Stiefel manifold is always contained in an orbit of the isotropy subgroup of the Grassmannian. In the special cases when the image is the full isotropy orbit, the metrics induced on the Stiefel manifold are shown to be among those metrics considered in [1], and it is shown that Einstein metrics occur among them. This briefly summarizes the content of Theorems A, B and C.

A formula is derived for the sectional curvature of these special metrics (Proposition 3), and it is used in the discussion of two sets of examples. It is shown that some of the Einstein metrics on the real Stiefel manifolds have both positive and negative sectional curvatures (Proposition 4). The pinching of the non-constantly curved Einstein metric on S^{4q+3} is computed (Proposition 5) and observed to be positive and approaching zero as q increases.

The author wishes to express thanks to E. Wilson for very helpful conversations about these imbeddings.

§1. Let $\mathbf{F}$ denote the field of real numbers $\mathbf{R}$, or complex numbers $\mathbf{C}$, or quaternions $\mathbf{H}$. Let $\mathbf{F}^{n\times p}$ denote the Euclidean space of all $n \times p$ matrices with entries in $\mathbf{F}$, where n and p are positive integers. Let $\mathbf{F}^{n\times p^*}$ denote the open submanifold consisting of all those matrices of maximal rank. The general linear group $GL(p; \mathbf{F})$ acts on $\mathbf{F}^{n\times p^*}$ on the right by ordinary matrix multiplication. When $n > p$, the Grassmannian $G_{n,p}$ of p-planes in $\mathbf{F}^n$ (i.e. p-dimensional $\mathbf{F}$-subspaces) is the orbit space $\mathbf{F}^{n\times p^*}/GL(p; \mathbf{F})$. Denote the natural projection $\mathbf{F}^{n\times p^*} \to G_{n,p}$ by π.

Let h denote the standard hermitian form on $\mathbf{F}^n$ given by $h(\xi, \eta) = \xi^*\eta$, $\xi, \eta \in \mathbf{F}^n$, where ξ^* denotes the conjugate transpose of the column vector ξ.

Received February 5, 1975. This work partially supported by N.S.F. contract number GP 34710.

The Stiefel manifold $S_{n,p}$ of unitary p-frames in $\mathbf{F}^n$ is the closed imbedded submanifold of $\mathbf{F}^{n\times p}$ defined by $S_{n,p} = \{Z \in \mathbf{F}^{n\times p} : Z^*Z = I_p\}$, where I_p is the $p \times p$ identity matrix. Of course $S_{n,p} \subset \mathbf{F}^{n\times p^*}$. Note that $S_{n,1}$ is the unit sphere in $\mathbf{F}^{n\times 1} = \mathbf{F}^n$.

Let $S(n)$ denote $O(n)$, $U(n)$, or $Sp(n)$ when $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, respectively. Thus $S(n) = \{A \in GL(n; \mathbf{F}) : A^*A = I_n\}$. Let $LS(n) = \{A \in \mathbf{F}^{n\times n} : A^* + A = 0\}$ denote the Lie algebra of $S(n)$.

$S(n)$ acts on the left by matrix multiplication on $\mathbf{F}^{n\times p^*}$ and preserves $S_{n,p}$, $0 < p < n$. We fix an origin in $S_{n,p}$ to be

$$Z_0 = \begin{pmatrix} I_p \\ 0 \end{pmatrix},$$

where 0 denotes the $n - p \times p$ zero matrix. The action of $S(n)$ on $S_{n,p}$ is transitive and the isotropy subgroup at Z_0 is $S(q)$, where $q = n - p$, contained in $S(n)$ by

$$S(q) \ni b \leftrightarrow \begin{pmatrix} I_p & 0 \\ 0 & b \end{pmatrix} \in S(n).$$

Thus $S_{n,p} \cong S(n)/S(q)$.

Since the left action of $S(n + p)$ on $\mathbf{F}^{n+p\times p^*}$ commutes with the right action of $GL(p; \mathbf{F})$ it induces a transitive left action of $S(n + p)$ on the Grassmannian $G_{n+p,p}$ whose isotropy subgroup at the origin

$$\pi\begin{pmatrix} I_p \\ 0 \end{pmatrix} \quad \text{is} \quad S(p) \times S(n),$$

which is contained in $S(n + p)$ by

$$S(p) \times S(n) \ni (a, b) \leftrightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in S(n + p).$$

Thus $G_{n+p,p} \cong S(n + p)/S(p) \times S(n)$.

Up to homothety, there is a unique $S(n)$-invariant Riemannian metric on $G_{n,p}$. This metric is induced on $G_{n,p}$ from the symmetric (0, 2)-tensor on $\mathbf{F}^{n\times p^*}$ given at the point $Z \in \mathbf{F}^{n\times p^*}$ by

$$d\tilde{s}^2 = \frac{4}{p} \operatorname{Re} \operatorname{Tr} (Z^*Z)^{-1}\{dZ^*\, dZ - dZ^*Z(Z^*Z)^{-1}Z^*\, dZ\},$$

where Tr denotes trace and Re denotes real part. This form is invariant under the right action of $GL(p; \mathbf{F})$, and therefore induces a form ds^2 on $G_{n,p}$. Furthermore, $d\tilde{s}^2$ is invariant under the left action of $S(n)$, which implies that ds^2 is the canonical Riemannian metric on $G_{n,p}$. It can be checked that the maximum sectional curvature of ds^2 on $G_{n,p}$ is equal to one.

Example. When $\mathbf{F} = \mathbf{C}$, $p = 1$, then $S_{n,1}$ is the unit $2n - 1$ sphere in $\mathbf{C}^n$, $G_{n+1,1}$ is the complex projective space $\mathbf{C}P^n$, and ds^2 is the Fubini-Study metric on $\mathbf{C}P^n$ of constant holomorphic sectional curvature equal to one.

For any $\Lambda \in \mathbf{F}^{p\times p}$ associate the map $\hat{f} : S_{n,p} \to \mathbf{F}^{n+p\times p^*}$ given by

$$\hat{f}(Z) = \begin{pmatrix} I_p \\ Z\Lambda \end{pmatrix}.$$

Then the differential of $\hat{f}$ is given by

$$\hat{f}_* X = \begin{pmatrix} 0 \\ XA \end{pmatrix}, \qquad X \in T_Z(S_{n,p}) \subseteq \mathbf{F}^{n\times p}.$$

Let $f = \pi \circ \hat{f} : S_{n,p} \to G_{n+p,p}$. Define

$$F: S(n) \to S(n+p) \quad \text{by} \quad F(A) = \begin{pmatrix} I_p & 0 \\ 0 & A \end{pmatrix}, \quad \text{and let} \quad Z_0 = \begin{pmatrix} I_p \\ 0 \end{pmatrix}$$

be a fixed origin of $S_{n,p}$.

THEOREM A.

1) *f is a C^ω map, which is $1:1$ and non-singular if and only if Λ is non-singular. Hence for any $\Lambda \in GL(p; \mathbf{F})$, the associated map f defines a C^ω imbedding of $S_{n,p}$ into $G_{n+p,p}$.*
2) *$f \circ A = F(A) \circ f$ for any $A \in S(n)$. Hence f^*ds^2 is $S(n)$-invariant.*
3) *If f_i is the map defined by Λ_i in $GL(p;\mathbf{F})$, $i = 1, 2$, then $f_1(S_{n,p}) \cap f_2(S_{n,p}) \neq \varnothing$ if and only if $\Lambda_1\Lambda_2^{-1} \in S(p)$, in which case $f_1 = f_2 \circ R_M$, where $M = \Lambda_1\Lambda_2^{-1} \in S(p)$ and $R_M : S_{n,p} \to S_{n,p}$ is defined by $R_M(Z) = ZM$. In particular, $f_1(S_{n,p}) \cap f_2(S_{n,p}) \neq \varnothing$ if and only if $f_1(S_{n,p}) = f_2(S_{n,p})$.*
4) *If $\Lambda \in GL(p; \mathbf{F})$, then $f(S_{n,p}) \subseteq S(p) \times S(n)$. X_0 = the orbit through X_0 of the isotropy subgroup of $G_{n+p,p}$, where $X_0 = f(Z_0)$. Equality holds if and only if $\Lambda^*\Lambda = \lambda I_p$ for some $\lambda > 0$.*
5) *If*

$$\Lambda = \lambda I_p, \qquad \lambda > 0, \qquad \begin{pmatrix} a \\ X \end{pmatrix}, \begin{pmatrix} b \\ Y \end{pmatrix} \in T_{Z_0}(S_{n,p}),$$

then

$$f^* ds^2\left(\begin{pmatrix} a \\ X \end{pmatrix}, \begin{pmatrix} b \\ Y \end{pmatrix}\right) = \frac{4}{p}\frac{\lambda^2}{1+\lambda^2} \operatorname{Re} \operatorname{Tr}\left\{X^*Y + \frac{1}{1+\lambda^2}\, a^*b\right\}.$$

Proof. 1), 2) and 5) are clear.

3) $f_1(S_{n,p}) \cap f_2(S_{n,p}) \neq \varnothing \Rightarrow$ there exist $Z, W \in S_{n,p}$ such that $f_1(Z) = f_2(W)$. But

$$f_1(Z) = \pi\begin{pmatrix} I_p \\ Z\Lambda_1 \end{pmatrix}, \qquad f_2(W) = \pi\begin{pmatrix} I_p \\ W\Lambda_2 \end{pmatrix}$$

so $f_1(Z) = f_2(W)$ implies $Z\Lambda_1 = W\Lambda_2$. Therefore

$$W = ZM, \quad \text{where we set } M = \Lambda_1\Lambda_2^{-1} \in GL(p; \mathbf{F}).$$

Now there exist $A, B \in S(n)$ such that $Z = AZ_0$, $W = BZ_0$. Thus $W = ZM$, and $A^{-1}BZ_0 = Z_0M$.

Since

$$A^{-1}B \in S(n) \quad \text{and} \quad Z_0 = \begin{pmatrix} I_p \\ 0 \end{pmatrix},$$

it follows that $M \in S(p)$. Hence $\Lambda_1 = M\Lambda_2$ for some $M \in S(p)$, and so $f_1 = f_2 \circ R_M$.

Conversely, if $\Lambda_1 = M\Lambda_2$ for some $M \in S(p)$, then $f_1 = f_2 \circ R_M$ and clearly $f_1(S_{n,p}) = f_2(S_{n,p})$, since $R_M : S_{n,p} \to S_{n,p}$ is a diffeomorphism.

4) Take $\Lambda \in GL(p; \mathbf{F})$, and let $Z \in S_{n,p}$. Then $Z = AZ_0$ for some $A \in S(n)$, and therefore

$$f(Z) = f \circ A(Z_0) = F(A) \circ f(Z_0) = F(A)X_0 \in S(p) \times S(n)\cdot X_0,$$

since $F(A) \in S(p) \times S(n)$. Thus $f(S_{n,p}) \subseteq S(p) \times S(n)\cdot X_0$.

Suppose equality holds. Then for every $A \in S(p)$, $B \in S(n)$, there exists a $Z \in S_{n,p}$ such that $f(Z) = (A, B)\cdot X_0$,

$$\pi\begin{pmatrix} I_p \\ Z\Lambda \end{pmatrix} = \pi\begin{pmatrix} A \\ BZ_0\Lambda \end{pmatrix} = \pi\begin{pmatrix} I_p \\ BZ_0\Lambda A^{-1} \end{pmatrix}.$$

Thus $Z = BZ_0\Lambda A^{-1}\Lambda^{-1} \in S_{n,p}$ for every $A \in S(p)$, $B \in S(n)$. Thus $I_p = \Lambda^{-1*}A^{-1*}\Lambda^* Z_0^* B^* BZ_0\Lambda A^{-1}\Lambda^{-1} = \Lambda^{*-1}A\Lambda^*\Lambda A^{-1}\Lambda^{-1}$, i.e. $\Lambda^*\Lambda = A\Lambda^*\Lambda A^{-1}$ for every $A \in S(p)$. Hence $\Lambda^*\Lambda$ is in the centeralizer of $S(p)$ in $GL(p; \mathbf{F})$, i.e. $\Lambda^*\Lambda = \lambda I_p$ for some $\lambda \in \mathbf{R}$ if $\mathbf{F} = \mathbf{R}$ or $\mathbf{H}$, or for some $\lambda \in \mathbf{C}$ if $\mathbf{F} = \mathbf{C}$. But $\Lambda^*\Lambda$ hermitian implies $\lambda \in \mathbf{R}$ in all cases, and $0 < \operatorname{Tr} \Lambda^*\Lambda = \lambda p$ implies $\lambda > 0$. This completes the proof of Theorem A.

If O_i denotes the $i \times i$ zero matrix, then

$$X = \begin{bmatrix} 0_p & -I_p & 0_q \\ I_p & & \\ & & 0_n \\ 0_q & & \end{bmatrix},$$

$n = p + q$, is in $LS(n + p)$, and the projection of the 1-parameter subgroup $\exp (t/2)X$ of $S(n + p)$ onto the quotient space $G_{n+p,p}$ is a geodesic with respect to ds^2. It is given by

$$\sigma(t) = \pi\begin{bmatrix} \cos \frac{t}{2} I_p \\ \sin \frac{t}{2} I_p \\ 0_q \end{bmatrix} = \pi\begin{bmatrix} I_p \\ \tan \frac{t}{2} I_p \\ 0_q \end{bmatrix}.$$

Since $ds^2(\dot\sigma, \dot\sigma) = 1$, it follows that t is arc-length parameter, for $0 \leq t \leq \pi$.

THEOREM B

Let f_Λ be the imbedding associated to $\Lambda \in GL(p; \mathbf{F})$, and let $\sigma(t)$, $0 \leq t \leq \pi$ be the above geodesic in $G_{n+p,p}$.

1) *If* $f_\Lambda(S_{n,p}) = S(p) \times S(n) \cdot f_\Lambda(Z_0)$, *then there exists a unique* r, $0 < r < \pi$, *such that* $f_\Lambda(S_{n,p}) = S(p) \times S(n) \cdot \sigma(r)$.
2) *Conversely, for each* r, $0 < r < \pi$, *if* $\lambda = \tan r/2$ *and* $\Lambda = \lambda I_p$ *then* $f_\Lambda(S_{n,p}) = S(p) \times S(n) \cdot \sigma(r)$. *In fact* $\sigma(r) = f_\Lambda(Z_0)$.

Proof.

1) By 4) of Theorem A, equality of the image with the whole isotropy orbit implies that $\Lambda^*\Lambda = \lambda^2 I_p$ for some $\lambda > 0$. Let $r = 2 \arctan \lambda$, $0 < r < \pi$, and let $\Lambda_2 = \lambda I_p$. Then

$$\Lambda_2{}^*\Lambda_2 = \lambda^2 I_p \quad \text{and} \quad f_{\Lambda_2}(Z_0) = \pi\begin{bmatrix} I_p \\ I_p\lambda \\ 0 \end{bmatrix} = \pi\begin{bmatrix} I_p \\ I_p \tan \frac{r}{2} \\ 0 \end{bmatrix} = \pi\begin{bmatrix} I_p \cos \frac{r}{2} \\ I_p \sin \frac{r}{2} \\ 0 \end{bmatrix} = \sigma(r),$$

so by 4) of Theorem A $f_{\Lambda_2}(S_{n,p}) = S(p) \times S(n) \cdot \sigma(r)$. But, if we let $M = \Lambda\Lambda_2{}^{-1} = \lambda^{-1}\Lambda$, then $M \in S(p)$ and so by 3) of Theorem A, $f_\Lambda(S_{n,p}) = f_{\Lambda_2}(S_{n,p})$.

To show uniqueness, suppose $S(p) \times S(n) \cdot \sigma(r_1) = S(p) \times S(n) \cdot \sigma(r_2)$, for some $0 < r_1 \leq r_2 < \pi$. Let $\lambda_i = \tan r_i/2$, and let $\Lambda_i = \lambda_i I_p$, for $i = 1, 2$. Then $f_{\Lambda_1}(S_{n,p}) = f_{\Lambda_2}(S_{n,p})$, so by 3) of Theorem A, $\lambda_1\lambda_2{}^{-1}I_p = \Lambda_1\Lambda_2{}^{-1} \in S(p)$. Hence $\lambda_1 = \lambda_2$.

2) was proved in the course of proving 1) above. This completes the proof of Theorem B.

§2. The metric induced on $S_{n,p}$. We consider the metric induced on $S_{n,p}$ from the imbedding given by matrices of the form $\Lambda = \lambda I_p$, $\lambda > 0$. Let $\lambda > 0$ and let $f = f_\Lambda$, where $\Lambda = \lambda I_p$. Then the $S(n)$-invariant metric on $S_{n,p}$ f^*ds^2 is given by 5) of Theorem A. Let $T = 1/(1 + \lambda^2)$, so $0 < T < 1$. Then

$$(*) \qquad f^*\, ds^2\left(\begin{pmatrix} a \\ X \end{pmatrix}, \begin{pmatrix} b \\ Y \end{pmatrix}\right) = \frac{4}{p}(1 - T)\, \mathrm{Re\ Tr}\, \{X^*Y + Ta^*b\},$$

where $a, b \in LS(p)$, $X, Y \in \mathbf{F}^{q\times p}$, $p + q = n$.

Consider the bi-invariant Riemannian metric on $S(n)$ given by $B(u, v) = \mathrm{Re\ Tr}\ u^*v$, where $u, v \in LS(n)$. (Note Tr u^*v is already real when $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$.) Then $LS(n) = V \oplus LS(p) \oplus LS(q)$ is a B-orthogonal direct sum, where $LS(p) \subseteq LS(n)$ as

$$\left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in LS(p)\right\} \quad \text{and} \quad LS(q) \subseteq LS(n) \quad \text{as} \quad \left\{\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} : b \in LS(q)\right\}$$

$$\text{and} \quad V = \left\{\begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix} : X \in \mathbf{F}^{q\times p}\right\}.$$

Proposition 1. *Regarding* f^*ds^2 *as a symmetric bilinear form on* $LS(n)$, *we have* $f^*ds^2 = (2/p)(1 - T)\{B|_{V\times V} + 2TB|_{LS(p)\times LS(p)}\}$, *that is*

$$f^*\,ds^2\left(\begin{pmatrix} a \\ X \end{pmatrix}, \begin{pmatrix} b \\ Y \end{pmatrix}\right)$$
$$= \frac{2}{p}(1 - T)\left\{B\left(\begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}, \begin{pmatrix} 0 & -Y^* \\ Y & 0 \end{pmatrix}\right) + 2TB\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\right)\right\},$$

where $X, Y \in \mathbf{F}^{q\times p}$, $a, b \in LS(p)$.

We are thus in the context of [1], p. 604, where the parameter t of that paper is related to T here by $T = t^2/2$.

We will need the Killing form F of $LS(n)$.

1) When $\mathbf{F} = \mathbf{R}$, $F(u, v) = (n - 2)\,\mathrm{Tr}\,uv$, for $u, v \in LS(n) = \mathbf{o}(n)$. Thus $F = -(n - 2)B$.
2) When $\mathbf{F} = \mathbf{C}$, $F(u, v) = 2n(\mathrm{Tr}\,uv - n^{-1}\,\mathrm{Tr}\,u\,\mathrm{Tr}\,v)$, for $u\,v \in LS(n) = \mathbf{u}(n)$. Recall that $\mathbf{u}(n) = \mathbf{su}(n) \oplus \mathbf{c}$, where $\mathbf{c}$ = center of $\mathbf{u}(n)$, and the sum is orthogonal with respect to either F or B. Then $F|_{\mathbf{c}\times\mathbf{c}} = 0$ and $F|_{\mathbf{su}(n)\times\mathbf{su}(n)} = -2nB|_{\mathbf{su}(n)\times\mathbf{su}(n)}$.
3) When $\mathbf{F} = \mathbf{H}$, $F(u, v) = 4(n + 1)\,\mathrm{Re}\,\mathrm{Tr}\,uv$, for $u, v \in LS(n) = \mathbf{sp}(n)$. Thus $F = -4(n + 1)B$.

PROPOSITION 2. *Let S_T denote the Ricci tensor of f^*ds^2, where $f = f_\Lambda$, $\Lambda = \lambda I_p$, $\lambda > 0$, $T = (1 + \lambda^2)^{-1}$. We regard S_T as a symmetric bilinear form on $LS(p) \oplus V$, which is naturally identified with the tangent space at Z_0 of $S_{n,p} = S(n)/S(q)$, $q = n - p$.*

1) $S_T|_{LS(p)\times V} = 0$
2) *When* $\mathbf{F} = \mathbf{R}$, $S_T|_{V\times V} = \frac{1}{2}(n - 2 - (p - 1)T)B|_{V\times V}$
$$S_T|_{\mathbf{o}(p)\times\mathbf{o}(p)} = (qT^2 + \tfrac{1}{4}(p - 2))B|_{\mathbf{o}(p)\times\mathbf{o}(p)}$$
3) *When* $\mathbf{F} = \mathbf{C}$, $S_T|_{V\times V} = (n - pT)B|_{V\times V}$. *Decomposing* $\mathbf{u}(p) = \mathbf{su}(p) \oplus \mathbf{c}$, *B-orthogonal direct sum as above,*
$$S_T|_{\mathbf{su}(p)\times\mathbf{c}} = 0$$
$$S_T|_{\mathbf{su}(p)\times\mathbf{su}(p)} = \left(Tq + \frac{p}{4T}\right)2TB|_{\mathbf{su}(p)\times\mathbf{su}(p)}$$
$$S_T|_{\mathbf{c}\times\mathbf{c}} = 2qT^2B|_{\mathbf{c}\times\mathbf{c}}\,.$$
4) *When* $\mathbf{F} = \mathbf{H}$, $S_T|_{V\times V} = (2(n + 1) - T(2p + 1))B|_{V\times V}$
$$S_T|_{\mathbf{sp}(p)\times\mathbf{sp}(p)} = (4T^2q + p + 1)B|_{\mathbf{sp}(p)\times\mathbf{sp}(p)}\,.$$

Proof. The Ricci tensor S_T of f^*ds^2 is given by equations (19), (20) and (21) on p. 609 of [1].

i) $S_T|_{V\times V} = S + T\alpha_1$, where $S = -\frac{1}{2}F|_{V\times V}$ by (12a) on p. 608 of [1] together with the fact that $[V, V] \subseteq LS(p) \oplus LS(q)$; and where α_1 is a symmetric bilinear form on V defined in the middle of p. 609 of [1] and which is invariant under the adjoint action of $S(p) \times S(q)$ on V.

ii) $S_T|_{V\times LS(p)} = -\frac{1}{2}TF|_{V\times LS(p)} = 0$, by the remarks preceding Proposition 2. This proves 1).

iii) $S_T|_{LS(p)\times LS(p)} = -T^2F|_{LS(p)\times LS(p)} + (T^2 - \frac{1}{4})F_1$, where F_1 is the Killing form of $LS(p)$.

The adjoint action of $S(p) \times S(q)$ on V is irreducible, and α_1 is invariant under it. Thus $\alpha_1 = aF|_{V\times V}$ for some real constant a, since F is invariant and negative definite on $V \times V$.

Let $\{v_i\}$, $i = 1, \cdots, m$, where $m = \dim V$, be a B-orthonormal basis of V and let $\{u_b\}$, $b = 1, \cdots, r$, where $r = \dim LS(p)$, be a B-orthonormal basis of $LS(p)$. By the lemma at the bottom of p. 610 of [1], (*) $\sum_b F(u_b, u_b) = \sum_i \alpha_1(v_i, v_i) + \sum_b F_1(u_b, u_b)$.

Case $\mathbf{F} = \mathbf{R}$. Now

$$m = pq, \qquad r = \tfrac{1}{2}p(p-1), \qquad F_1 = \frac{p-2}{n-2}F|_{\mathbf{o}(p)\times\mathbf{o}(p)}.$$

The second equation of 2) follows upon substitution of this expression for F_1 into iii). Thus, for each b, $F(u_b, u_b) = -(n-2)$ and $F_1(u_b, u_b) = -(p-2)$, and for each i, $\alpha_1(v_i, v_i) = aF(v_i, v_i) = -a(n-2)$. By (*), $-(n-2)\frac{1}{2}p(p-1) = -a(n-2)pq - (p-2)\frac{1}{2}p(p-1)$, and therefore $a = \frac{1}{2}(p-1)/(n-2)$. The first equation of 2) follows upon substitution of this value of a into i).

Case $\mathbf{F} = \mathbf{H}$. Now

$$m = 4pq, \qquad r = p(2p+1), \qquad F_1 = \frac{p+1}{n+1}F|_{\mathbf{sp}(p)\times\mathbf{sp}(p)}.$$

The second equation of 4) follows upon substitution of this expression for F_1 into iii). Using (*) as in the case $\mathbf{F} = \mathbf{R}$, it follows that $a = (2p+1)/4(n+1)$. Substituting this value of a into i) gives the first equation of 4).

Case $\mathbf{F} = \mathbf{C}$. Now $m = 2pq$, $r = p^2$, but F_1 is not a multiple of $F|_{\mathbf{u}(p)\times\mathbf{u}(p)}$. We may assume the B-orthonormal basis $\{u_b\}$ of $\mathbf{u}(p)$ to have been chosen such that $u_1 \in \mathbf{c}$ and $u_b \in \mathbf{su}(p)$, $b > 1$, since the direct sum $\mathbf{u}(p) = \mathbf{c} \oplus \mathbf{su}(p)$ is B-orthogonal. (Here $\mathbf{c}$ = center of $\mathbf{u}(p)$.) Then

$$u_1 = \begin{pmatrix} siI_p & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad s = p^{-\frac{1}{2}} \quad \text{and} \quad i = \sqrt{-1},$$

and therefore $F(u_1, u_1) = 2n(\operatorname{Tr} u_1u_1 - n^{-1}\operatorname{Tr} u_1 \operatorname{Tr} u_1) = -2s^2pq = -2q$, while $F_1(u_1, u_1) = 0$. For $b > 1$, $u_b \in \mathbf{su}(p)$ and $F(u_b, u_b) = -2n$ while $F_1(u_b, u_b) = -2p$. Finally, $F(v_j, v_j) = -2n$ since each $v_j \in \mathbf{su}(n)$. Hence by (*), $-2q - 2n(p^2-1) = -2na2pq - 2p(p^2-1)$. Solving, we get $a = p/2n$.

Substituting this value of a into i) gives the first equation of 3). The second equation of 3) follows from iii) and the fact that the direct sum $\mathbf{u}(p) = \mathbf{c} \oplus \mathbf{su}(p)$ is orthogonal with respect to both F and F_1.

We have just shown above that

$$F|_{\mathbf{c}\times\mathbf{c}} = -2qB|_{\mathbf{c}\times\mathbf{c}}, \; F|_{\mathbf{su}(p)\times\mathbf{su}(p)} = -2nB|_{\mathbf{su}(p)\times\mathbf{su}(p)}, \; F_1|_{\mathbf{c}\times\mathbf{c}} = 0,$$

and $F_1|_{su(p)\times su(p)} = -2pB|_{su(p)\times su(p)}$, (where $\mathbf{c}$ is the center of $\mathbf{u}(p)$). Now the third and fourth equations of 3) follow from substitution into iii).

This completes the proof of Proposition 2.

Combining Propositions 1 and 2 gives

THEOREM C. *Continue to let S_T denote the Ricci tensor of f^*ds^2, where $f = f_\Lambda$, $\Lambda = \lambda I_p$, $\lambda > 0$, $T = (1 + \lambda^2)^{-1}$. Then the following holds.*

Case $\mathbf{F} = \mathbf{R}$: f^*ds^2 is an Einstein metric, i.e. *S_T is a constant multiple of f^*ds^2, if and only if T satisfies $(n - 1)T^2 - (n - 2)T + \frac{1}{4}(p - 2) = 0$. This equation has two solutions, T_- and T_+ , satisfying $0 \leq T_- < T_+ < 1$, with $T_- = 0$ occurring if and only if $p = 2$. Furthermore, for fixed p, $T_- \to 0$ and $T_+ \to 1$ as $n \to \infty$.*

Case $\mathbf{F} = \mathbf{C}$: f^*ds^2 is never an Einstein metric.

Case $\mathbf{F} = \mathbf{H}$: f^*ds^2 is an Einstein metric if and only if T satisfies $2(2n + 1)T^2 - 4(n + 1)T + p + 1 = 0$. This equation has two solutions, T_- and T_+ , satisfying $0 < T_- < T_+ \leq 1$, with $T_+ = 1$ occurring if and only if $p = 1$. Furthermore, for fixed p, $T_- \to 0$ and $T_+ \to 1$ as $n \to \infty$.

Remarks. (1) In the case $\mathbf{F} = \mathbf{C}$, $p = 1$, the limiting value of $f^*ds^2 = 2(1 - T)\{B|_{V\times V} + 2TB|_{u(1)\times u(1)}\}$ as $T \to 1$ is the standard metric of constant curvature on $S_{n,1} = S^{2n-1}$. To be precise, $B|_{V\times V} + 2B|_{u(1)\times u(1)}$ defines the constant curvature metric on $U(n)/U(n - 1)$.

(2) In the case $\mathbf{F} = \mathbf{H}$, $p = 1$, the limiting value of f^*ds^2 as $T \to 1$ is the standard metric of constant curvature on $S^{4n-1} = Sp(n)/Sp(n - 1)$, in the same sense as in Remark (1). We shall discuss this example in greater detail below.

§3. In this section we shall compute the sectional curvature of $f^* ds^2$. In the following, for any Lie group G, $L(G)$ denotes its Lie algebra of left-invariant vector fields.

Let K be a compact Lie group with bi-invariant Riemannian metric B, let H be a closed subgroup and suppose $L(H) = L(H_1) \oplus L(H_2)$ is a B-orthogonal direct sum of ideals. Let V be the orthogonal complement in $L(K)$ of $L(H)$. Let T be a fixed positive number, and define an $ad(H)$-invariant inner product g on $V \oplus L(H_1)$ by $g \mid V \times L(H_1) = 0$, $g \mid V \times V = B \mid V \times V$ and $g \mid L(H_1) \times L(H_1) = 2TB \mid L(H_1) \times L(H_1)$. Then g defines a K-invariant Riemannian metric on K/H_2 .

PROPOSITION 3.

1) *The Levi-Civita connection of g is given by the linear map $\Lambda : L(H_1) \oplus V \to$ End $(L(H_1) \oplus V)$, defined for $a, b \in L(H_1)$, $X, Y \in V$ by*

i) $\Lambda(X)Y = \frac{1}{2}[X, Y]_{H_1}$ (= *$L(H_1)$-component of* $[X, Y]$).

ii) $\Lambda(a)Y = (1 - T)[a, Y]$

iii) $\Lambda(X)b = T[X, b]$

iv) $\Lambda(a)b = \frac{1}{2}[a, b]$

2) *Suppose that K/H is symmetric,* i.e. $[V, V] \subseteq L(H)$. *Then the curvature*

of g on K/H_2, regarded as a quadrilinear form on $L(H_1) \oplus V$, is given for $E = a + X$, $F = b + Y$, $a, b \in L(H_1)$, $X, Y \in V$ by:

$$R(E, F, E, F) = \|[X, Y]\|^2 - \tfrac{3}{2}T \|[X, Y]_{H_1}\|^2 + T(3 - 2T)B([X, Y], [a, b])$$
$$+ \frac{T}{2} \|[a, b]\|^2 + T^2 \|[a, Y]\|^2 + T^2 \|[b, X]\|^2 + 2T^2B([a, X], [b, Y]),$$

where the norm $\|\cdot\|$ refers to B, i.e. $\|Z\|^2 = B(Z, Z)$.

Remark. The Ricci tensor of g is given by Proposition 11 in [1].

Proof of Proposition 3.

1) Λ is characterized by the two properties that $\Lambda(E)$ be skew-adjoint with respect to g and that $\Lambda(E)F - \Lambda(F)E = [E, F]_{V \oplus H_1} = V \oplus L(H_1)$-component of $[E, F]$, for all $E, F \in V \oplus L(H_1)$. These properties are easily verified for Λ defined in the proposition.

2) This is a routine computation, Cf [2]. Proposition 2.3, p. 191.

Applications.

The real Stiefel manifolds $S_{n,p} = O(n)/O(n - p)$.

These spaces provide examples of compact Einstein spaces whose sectional curvature takes both positive and negative values.

Let $n = p + q$, with $p, q \geq 2$. Now $K = O(n)$, $H_1 = O(p)$,

$$V = \left\{\begin{pmatrix} 0 & -t^{+}\xi \\ \xi & 0_q \end{pmatrix} : \xi \in \mathbf{R}^{q \times p}\right\}, \qquad L(H_1) = \left\{\begin{pmatrix} a & 0 \\ 0 & 0_q \end{pmatrix} : a \in \mathfrak{o}(p)\right\},$$

and for the bi-invariant metric on $\mathfrak{o}(n)$ we take the B of Proposition 1, namely $B(X, Y) = \mathrm{Tr}^t XY$. Then for $T > 0$, the metric g of Proposition 3 is $g((a, \xi), (b, \eta)) = 2(\langle \xi, \eta\rangle + T\langle a, b\rangle)$, where $\xi, \eta \in \mathbf{R}^{q \times p}$, $a, b \in \mathfrak{o}(p)$, $\langle \xi, \eta \rangle = \mathrm{Tr}^t\, \xi\eta$, $\langle a, b\rangle = \mathrm{Tr}^t\, ab$. Thus, if $0 < T < 1$, then

$$f^* \, ds^2 = \frac{2}{p}(1 - T)g, \quad \text{where} \quad f = f_\Lambda, \qquad \Lambda = \lambda I_p, \qquad T = (1 + \lambda^2)^{-1}.$$

Applying the formula in Proposition 3 we get the curvature of g for $E = (a, \xi)$, $F = (b, \eta)$ to be:

$$(R(E, F, E, F) = |\eta^t\xi - \xi^t\eta|^2 + (1 - \tfrac{3}{2}T)|\,{}^t\eta\xi - {}^t\xi\eta|^2 + \frac{T}{2}\|[a, b]\|^2$$
$$+ 2T^2(|\eta a|^2 + |\xi b|^2) + 2T(3 - 4T)\langle \xi a, \eta b\rangle - 2T(3 - 2T)\langle \xi b, \eta a\rangle,$$

where for any matrix A, $|A|^2 = \mathrm{Tr}^t\, AA$.

By Theorem C, g is an Einstein metric when

$$T = T_+ = \tfrac{1}{2}(n - 1)^{-1}\{n - 2 + [(n - 2)^2 - (n - 1)(p - 2)]^{\frac{1}{2}}\}, \text{ and } 0 < T_+ < 1.$$

Designate this Einstein metric by g_+.

PROPOSITION 4. *When $q \geq p > 2$, the Einstein metric g_+ has sectional curvature of both positive and negative values.*

Proof. Let P be a plane in $\mathfrak{o}(p)$ with g_+-orthonormal basis a, b such that $[a, b] \neq 0$. Then the sectional curvature of P is $K(P) = \frac{1}{2}T_+ \,|[a, b]|^2 > 0$.

On the other hand, if E_{ij} denotes the $q \times p$ matrix with 1 in the ijth entry, 0 elsewhere, and if $\xi = E_{11}$, $\eta = E_{12}$, then $K(\xi \wedge \eta) = \frac{1}{4}(2 - 3T_+)$, which is easily shown to be negative when $n = p + q > 4$ and $p \leq \frac{1}{2}n$.

Distance spheres in quaternionic projective space.

We omit the details for the general quaternionic Stiefel manifolds and consider only the interesting case when $p = 1$, i.e. the unit sphere in $\mathbf{H}^n$.

Now $K = \mathrm{Sp}(q + 1)$, $H = \mathrm{Sp}(1) \times \mathrm{Sp}(q)$, $H_1 = \mathrm{Sp}(1)$, $H_2 = \mathrm{Sp}(q)$,

$$V = \left\{\begin{pmatrix} 0 & -\xi^* \\ \xi & 0_q \end{pmatrix} : \xi \in \mathbf{H}^{q\times 1} = \mathbf{H}^q\right\},$$

$$L(H_1) = \mathbf{sp}(1) = \left\{\begin{pmatrix} a & 0 \\ a & 0_q \end{pmatrix} : a \in \mathbf{H},\, \bar{a} = -a\right\},$$

and the bi-invariant metric on $\mathbf{sp}(q + 1)$ is $B(X, Y) = \mathrm{Re\,Tr}\, X^*Y$. (Here X^* denotes the conjugate transpose of X.) In the following ξ, $\eta \in \mathbf{H}^q$ (column vectors), a, $b \in \mathbf{sp}(1)$ (purely imaginary quaternions), h denotes the hermitian form on $\mathbf{H}^q$ given by $h(\xi, \eta) = \xi^*\eta$, and $\langle \xi, \eta\rangle = \mathrm{Re}\, h(\xi, \eta)$. Elements of $\mathbf{sp}(1) \oplus V$ are denoted by pairs (a, ξ), i.e.

$$(a, \xi) = \begin{pmatrix} a & -\xi^* \\ \xi & 0_q \end{pmatrix}.$$

For $T > 0$, the metric g of Proposition 3 is $g((a, \xi), (b, \eta)) = 2(\langle \xi, \eta\rangle + T\langle a, b\rangle)$. When $0 < T < 1$, it follows from Proposition 1 that $f^*ds^2 = 4(1 - T)g$, where $f = f_\Lambda$, $\Lambda = \lambda 1_p$, $\lambda > 0$, and $T = (1 + \lambda^2)^{-1}$.

Applying the formula in Proposition 3 we get the curvature of g for $E = (a, \xi)$, $F = (b, \eta)$ to be:

$$\begin{aligned}\tfrac{1}{2}R(E, F, E, F) = {} & |\xi|^2\,|\eta|^2 - \langle \xi, \eta\rangle^2 + T\,|\mathrm{Im}\, ab|^2 \\ & + T^2(|a|^2\,|\eta|^2 + |b|^2\,|\xi|^2) - 2T\langle \xi a, \eta b\rangle \\ & + 3(1 - T)\,|\mathrm{Im}\, h(\eta, \xi)|^2 + 2T(3 - 2T)\langle \mathrm{Im}\, h(\eta, \xi), \mathrm{Im}\, ab\rangle,\end{aligned}$$

where absolute value signs denote the norm defined by h.

If P is any plane in $\mathbf{sp}(1) \oplus V$, it is easily shown that P has a g-orthogonal basis $E = (a, \xi)$, $F = (b, \eta)$ such that $\langle a, b\rangle = 0 = \langle \xi, \eta\rangle$. Using such a basis, the sectional curvature of P is:

$$K(P) = \frac{|\xi|^2\,|\eta|^2 + T\,|ab|^2 + T^2(|a|^2\,|\eta|^2 + |b|^2\,|\xi|^2) + 3(1 - T)\,|h(\eta, \xi)|^2 + 6T(1 - T)\langle h(\eta, \xi), ab\rangle}{2(|\xi|^2 + T\,|a|^2)(|\eta|^2 + T\,|b|^2)}.$$

Let $K_{\min}$ and $K_{\max}$ denote the minimum and maximum values, respectively, of the sectional curvature of g (for fixed T, $0 < T \leq 1$). From a lengthy computation (made with considerable assistance from J. I. Hano) it follows that:

$$K_{\min} = \begin{cases} \frac{1}{2}T & \text{if } 0 < T \leq \frac{4}{5} \\ 2(4T^2 - 2T + 1)(11T + 1)^{-1} & \text{if } \frac{4}{5} \leq T \leq 1. \end{cases}$$

$$K_{\max} = \begin{cases} \frac{1}{2}T^{-1} & \text{if } 0 < T \leq \frac{1}{3} \\ \frac{1}{2}(4 - 3T) & \text{if } \frac{1}{3} < T \leq 1. \end{cases}$$

PROPOSITION 5.

The pinching of g (or f^*ds^2) *is:*

$$\delta = \frac{K_{\min}}{K_{\max}} = \begin{cases} T^2 & \text{if } 0 < T \leq \frac{1}{3} \\ T(4 - 3T)^{-1} & \text{if } \frac{1}{3} \leq T \leq \frac{4}{5} \\ 4(4T^2 - 2T + 1)(11T + 1)^{-1}(4 - 3T)^{-1} & \text{if } \frac{4}{5} \leq T \leq 1. \end{cases}$$

By Theorem C, g is an Einstein metric when T is $T_- = (2q + 3)^{-1}$ or $T_+ = 1$. Thus g_{T_+} is just the metric of constant positive curvature $(=\frac{1}{2})$ on the sphere $\mathrm{Sp}(q + 1)/\mathrm{Sp}(q)$, and g_{T_-} is an Eienstein metric on this sphere with positive sectional curvature with pinching $\delta = (2q + 3)^{-1}$. This example illustrates a rather weak relationship between sectional curvature and the Ricci tensor. To put it one way, for any $\epsilon > 0$, S^{4q+3} has an Einstein metric with pinching δ satisfying $0 < \delta < \epsilon$, for all sufficiently large q.

REFERENCES

1. G. R. JENSEN, *Einstein metrics on principal fibre bundles*, Journ. of Diff. Geom. **8** (1973), 599–614.
2. S. KOBAYASHI AND K. NOMIZU, *Foundations of differential geometry*, Vol. II, Interscience, New York, 1969.
3. A. WEINSTEIN, *Distance spheres in complex projective spaces*, Proc. Amer. Math. Soc. **39** (1973), 649–650.

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY, ST. LOUIS, MISSOURI 63130