SEMINAIRE DE GEOMETRIE DIFFERENTIELLE

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MINIMAL SURFACES IN SPHERES BY THE METHOD OF MOVING FRAMES

I. Introduction.-

These notes consist of an exposition of a large portion of the known results concerning minimal surfaces in spheres. Our purpose was not only to gather together these results, but also to derive them all from a uniform point of view, namely, by the method of moving frames. We used many sources of references, but the papers we used by far the most extensively were Bryant [2] and Chern [4], [5] .

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2. The moving frame method.-

Let N, g be on n-dimensional riemannian manifold. Let $\theta = \begin{bmatrix} \theta^1 \\ \theta^n \end{bmatrix} = (\theta^A)$ be an orthonormal coframe on an open set $U \subset N$, where $1 \leq A,B,C \leq n$, i.e. θ is a moving coframe and $g = \overset{n}{\Sigma} (\theta^A)^2$.

The Levi-Civita 1-forms of g with respect to θ are the 1-forms ω^A_B uniquely defined on U by

$$(2.1) \qquad d\theta^A = - \omega^A_B \wedge \theta^B \quad , \text{ and } \quad \omega^A_B = - \omega^B_A$$

(we'll always use Einstein's summation convention).

Geometrically the first equation points out the absence of torsion and the second the invariance of g under parallel transport.

The curvature forms with respect to θ are the 2-forms :

$$(2.2) \qquad \Omega^A_B = d\omega^A_B + \omega^A_C \wedge \omega^C_B \quad .$$

N, g is a space of constant curvature c if and only if

$$(2.3) \qquad \Omega^A_B = c \, \theta^A \wedge \theta^B \quad .$$

Let $f : M \to N$ be an immersion of a surface (i.e. $\dim M = 2$) in N . The induced metric on M is $f^*g = \sum_1^n (f^* \theta^A)^2$.

We say that the orthonormal coframe θ is a Darboux coframe along f if $f^* \theta^\alpha = 0$, where always $3 \leq \alpha, \beta, \gamma \leq n$ and $1 \leq i, j, k \leq 2$.

If θ is a Darboux coframe, we have :

$$0 = d(f^* \theta^\alpha) = - f^* \omega^\alpha_i \wedge f^* \theta^i \quad .$$

To simplify notation from now on we drop f^*. Then from Cartan's lemma :

(2.4) $$\omega_i^\alpha = h_{ij}^\alpha \theta^j \quad , \text{ where } h_{ij}^\alpha = h_{ji}^\alpha \quad , \text{ functions on } U .$$

The second fundamental forms with respect to θ are

(2.5) $$II^\alpha = \omega_i^\alpha \theta^i = h_{ij}^\alpha \theta^i \theta^j \quad \text{(symmetric product)} .$$

The second fundamental tensor II is defined by

(2.6) $$II = h_{ij}^\alpha e_\alpha \otimes \theta^i \theta^j \quad (e_\alpha \text{ are defined below}) .$$

The mean curvatures with respect to θ are

$$H^\alpha = \frac{1}{2}(h_{11}^\alpha + h_{22}^\alpha)$$

and the mean curvature vector

(2.7) $$H = H^\alpha e_\alpha$$

f is minimal iff $H = 0$; i.e. $H^\alpha = 0 \;\; \forall \alpha$.

In the case where N has constant curvature c, the Gaussian curvature of the induced metric $ds^2 = (\theta^1)^2 + (\theta^1)^2$ is

$$K = c + \sum_\alpha \det h^\alpha \quad , \text{ where } h^\alpha = (h_{ij}^\alpha) \quad .$$

From the dual point of view for each orthonormal coframe θ on $U \subset N$ we have a dual frame $e = (e_1, \ldots, e_n)$ where each e_A is a vector field on U, of length 1 , and

$$\theta^A(e_B) = \delta_B^A \quad .$$

The Levi-Civita covariant derivative $\nabla : \Gamma(TN) \to \Gamma(TN \otimes T^*N)$ is defined by :

(2.8) $$\nabla e_A = e_B \otimes \omega_A^B \quad .$$

If we consider the normal bundle $TM^\perp$ of M, then the vectors e_α form a local basis of $TM^\perp$, (for a Darboux frame) .

We can define a covariant derivative in the normal bundle $D : \Gamma(TM^\perp) \to \Gamma(TM^\perp \otimes T^*M)$ by setting :

(2.9) $$De_\alpha = e_\beta \otimes \omega_\alpha^\beta \quad ;$$

and if

(2.10) $$A_{e_\alpha} = - e_i \otimes \omega_\alpha^i = h_{ij}^\alpha \, e_i \otimes \theta^j$$

we get a $C^\infty(M)$-linear transformation $\Gamma(TM) \to \Gamma(TM)$ to be called the second fundamental tensor associated to the second fundamental form II^α .

From (2.8) we obtain :

(2.11) $$\nabla e_\alpha = - A_{e_\alpha} + De_\alpha$$

(e_α restricted to M) , which are the Weingarten equations.

Considering the structure equation (2.2) we get :

$$d\,\omega_\beta^\alpha = - \omega_\gamma^\alpha \wedge \omega_\beta^\gamma - \omega_i^\alpha \wedge \omega_\beta^i + \Omega_\beta^\alpha \quad ,$$

and defining the normal curvature forms

(2.12) $${}^\perp\Omega_\beta^\alpha = d\,\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma \quad ,$$

we have

$$^{\perp}\Omega^{\alpha}_{\beta} = \Omega^{\alpha}_{\beta} - \omega^{\alpha}_{i} \wedge \omega^{i}_{\beta}$$

and by (2.4)

$$(2.13) \qquad ^{\perp}\Omega^{\alpha}_{\beta} = \Omega^{\alpha}_{\beta} + h^{\alpha}_{ij}\, h^{\beta}_{ik}\, \theta^{j} \wedge \theta^{k} \quad .$$

In particular if N has constant curvature

$$(2.14) \qquad ^{\perp}\Omega^{\alpha}_{\beta} = h^{\alpha}_{ij}\, h^{\beta}_{ik}\, \theta^{j} \wedge \theta^{k} \quad .$$

Set

$$(2.15) \qquad ^{\perp}\Omega^{\alpha}_{\beta} = \frac{1}{2}\, {^{\perp}R}_{\alpha\beta ij}\, \theta^{i} \wedge \theta^{j} \quad .$$

The ${^{\perp}R}_{\alpha\beta ij}$ are the components of the normal curvature tensor. If R_{ABCD} are the components of the curvature tensor of N from (2.13) we have

$$(2.16) \qquad {^{\perp}R}_{\alpha\beta ij} = R_{\alpha\beta ij} + h^{\alpha}_{ki}\, h^{\beta}_{kj} - h^{\alpha}_{kj}\, k^{\beta}_{ki}$$

and in the case N has constant curvature :

$$(2.17) \qquad {^{\perp}R}_{\alpha\beta ij} = h^{\alpha}_{ki}\, h^{\beta}_{kj} - h^{\alpha}_{kj}\, h^{\beta}_{ki}$$

which are the Ricci equations.

It's important to know what happens under a change of Darboux frame. This is given by

$$\tilde{e} = e\, K$$

where $K = \begin{pmatrix} A & o \\ o & B \end{pmatrix}$, $A : U \cap \tilde{U} \to O(2)$, $B : U \cap \tilde{U} \to O(n-2)$

($O(n)$ the orthogonal group). From now on quantities with tilde refer to the new frame $\tilde{e}$.

Then

$$\theta = K\,\tilde{\theta} \text{ , i.e. } (\theta^i) = A(\tilde{\theta}^j) \text{ , } (\theta^\alpha) = B(\tilde{\theta}^\beta) \text{ .}$$

Taking exterior differential we have :

$$\tilde{\omega} = K^{-1}\,\omega\,K + K^{-1}\,dK \tag{2.18}$$

from which

$$\tilde{\omega}^\alpha_i = (B^{-1})^\alpha_\beta \quad \omega^\beta_j \quad A^j_i \qquad \text{and}$$

$$\tilde{h}^\alpha_{ij} = (B^{-1})^\alpha_\beta\, h^\beta_{\ell k}\, A^k_j\, A^\ell_i \text{ .} \tag{2.19}$$

Remark.

From (2.19) we have

$$\tilde{H}^\alpha = \frac{1}{2}\,(\tilde{h}^\alpha_{11} + \tilde{h}^\alpha_{22}) = (B^{-1})^\alpha_\beta\, H^\beta$$

and

$$\tilde{II}^\alpha = \tilde{\omega}^\alpha_i \quad \tilde{\theta}^i = (B^{-1})^\alpha_\beta\,\omega^\beta_j\,\theta^j = (B^{-1})^\alpha_\beta\, II^\beta \text{ .}$$

Now if M and N are orientable, we have a family of Darboux frames $\{e, U\}$ such that the open sets U give an open covering of M , and when $U \cap \tilde{U} \neq \emptyset$, $\tilde{e} = e\,K$, where $K = \begin{pmatrix} A & o \\ o & B \end{pmatrix}$, $A : U \cap \tilde{U} \to SO(2)$, $B : U \cap \tilde{U} \to SO(n-2)$.

In particular in the case $n = 3$, $B = 1$ and from the last two equations, H^3 and II^3 are globally defined on M .

Finally we consider the case $N = S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, with the canonical metric g induced by $\mathbb{R}^{n+1}$.
For each point $x \in S^n$, the tangent space of S^n at x can be identified with a subspace of $\mathbb{R}^{n+1}$, namely

$$T_x S^n = \{y \in \mathbb{R}^{n+1} : < y,x > = 0\}$$

(where $< , >$ is the canonical interior product in $\mathbb{R}^{n+1}$) .
Using this identification, we can consider each vector e_A of an orthonormal coframe $e = (e_1,\ldots,e_n)$ on an open set $U \subset S^n$ as a map $e_A : U \to \mathbb{R}^{n+1}$.
If, moreover, $x : U \to \mathbb{R}^{n+1}$ is the inclusion map, at each point of U , $x, e_1,\ldots,e_n$ is an orthonormal basis of $\mathbb{R}^{n+1}$. Then

$$\begin{aligned} dx &= e_A \otimes \theta^A \\ de_A &= e_B \otimes \omega_A^B + x \otimes \omega_A^o \end{aligned} \tag{2.20}$$

where θ^A , ω_B^A , ω_A^o are 1-forms on U defined by :

$$\begin{aligned} \theta^A &= <dx, e_A> \\ \omega_A^B &= <de_A, e_B> \\ \omega_A^o &= <de_A, x> \end{aligned} .$$

Taking exterior differential of (2.20) and of the equation

$$<e_A , e_B> = \delta_{AB}$$

we obtain :

$$\omega_A^o = -\theta^A$$
$$(2.21) \qquad d\theta^A = -\omega_B^A \wedge \theta^B \quad , \quad \omega_B^A = -\omega_A^B$$
$$d\omega_B^A = -\omega_C^A \wedge \omega_B^C + \theta^A \wedge \theta^B$$

Then $\theta = \begin{pmatrix}\theta^1 \\ \vdots \\ \theta^n\end{pmatrix}$ is the dual coframe and the ω_B^A are the Levi-Civita connection forms with respect to θ .

Moreover from the third of (2.21) we see that the curvature of this metric on S^n is constant equal to 1 .

We can also consider an orthonormal frame on $U \subset S^n$ from a more abstract point of view. Writing vectors of $\mathbb{R}^{n+1}$ as columns, we have a map

$$u = (x, e_1, \ldots, e_n) : U \to O(n+1)$$

and the pull-back via u of the Maurer-Cartan form of $O(n+1)$ is

$$u^{-1}\, du = \begin{vmatrix} 0 & -\theta^1, \ldots, -\theta^n \\ \theta^1 & \\ \vdots & \omega_B^A \\ \theta^n & \end{vmatrix} = \varphi \quad .$$

Equations (2.20) are simply $du = u \otimes \varphi$ and equations (2.21) are the structure equations of $O(n+1)$.

3. A result of H. Hopf.

In this paragraph we consider $N = S^3 = \{x \in \mathbb{R}^4 : |x| = 1\}$ with its canonical metric g of constant curvature 1.

A point $p \in M$ is called umbilical if $II_p^3 = \lambda\, ds_p^2$ for some $\lambda \in \mathbb{R}$, and it's easily seen that $\lambda = H^3(p) = H(p)$.

Of course M is totally umbilical if all of its points are umbilical.

There is an analogous result of a well known one for surfaces in $\mathbb{R}^3$.

Lemma.-

If $f : M \to S^3$ is a totally umbilical connected surface, then $f(M)$ is contained in a hyperplane section of S^3 ; i.e. $f(M) \subset S^3 \cap \Pi$, where Π is a hyperplane of $\mathbb{R}^4$.

Proof.

Let $e = (e_1, e_2, e_3)$ be a Darboux frame on U , θ its dual coframe. Then, f totally umbilical means

$$\omega_i^3 = H\,\theta^i \quad .$$

Taking exterior differential and using the structure equations (2.1), (2.2) we have

$$dH \wedge \theta^i = 0$$

i.e. H is constant on U . The case $H = 0$ is clear. Suppose $H \neq 0$.

Using equations (2.20), (2.21) we have

$$d(e_3 + Hf) = 0$$

i.e. $e_3 + Hf = n_0$ constant on U , or

(3.1) $$f = \frac{1}{H}(n_0 - e_3) \quad \text{on} \quad U \quad .$$

If we fix a point $p_0 \in U$, from (3.1) we deduce

$$< f - f(p_0), n_0 > = 0 \quad \text{on} \quad U$$

which means that $f(U)$ belongs to the hyperplane Π normal to n_0 and passing through $f(p_0)$. Since M is connected $f(M) \subset S^3 \cap \Pi$.

The following is due to H. Hopf [11] :

Theorem.-

Let $f : S^2 \to S^3$ be an immersion such that H is constant. Then $f(S^2)$ is contained in a hyperplane section of S^3 . In particular, if $f(S^2)$ is minimal, then $f(S^2)$ is an S^2-equator in S^3 .

Proof.

Let $e = (e_1, e_2, e_3)$ be an oriented Darboux frame on $U \subset S^3$, and let θ be its dual coframe. If we set

$$\varphi = \theta^1 + i\,\theta^2 \quad , \quad i = \sqrt{-1}$$

then φ defines an almost complex structure on S^2 (φ is of type (1,0)) which is always integrable (Korn-Lichtenstein Theorem). We have $ds^2 = \varphi\bar{\varphi}$.

We define the following linear map :

$$L : \mathcal{L}_2 \to \mathbb{C} \quad , \quad L(m_{ij}) = \frac{1}{2}(m_{11} - m_{22}) - im_{12}$$

where $\mathcal{L}_2$ is the vector space of 2×2 symmetric real matrices. If

$$A = \begin{vmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{vmatrix} \in SO(2)$$

we have

$$L({}^tAmA) = e^{2it} L(m) .$$

We set $\Psi = L(h)\,\varphi^2$ (where $h = (h^3_{ij})$)

a quadratic complex form of type (2,0) on U .

Lemma.-

If $f : M \to N^3$ is an oriented surface, then Ψ is globally defined on M. If N^3 is a space of constant sectional curvature, and H is constant, then Ψ is holomorphic.

Proof.

Let $\tilde{e} = e \cdot K$ be a change of oriented Darboux frames, where

$$K = \begin{pmatrix} A & o \\ o & 1 \end{pmatrix} \qquad A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \; : \; U \cap \tilde{U} \to SO(2) \; .$$

Then $\tilde{h} = {}^tAhA$ and therefore $L(\tilde{h}) = e^{2it} L(h)$, while $\tilde{\varphi} = e^{-it} \varphi$.
Therefore

$$L(\tilde{h})\, \tilde{\varphi}^2 = L(h)\, \varphi^2 \quad \text{on} \quad U \cap \tilde{U}$$

and Ψ is globally defined.

We use now the fact that N has constant curvature.

If z is a complex coordinate in a neighborhood of a point of U, we have $\varphi = b\,dz$, where b is a non-vanishing function on this neighborhood.
Then

$$\Psi = L(h)\, b^2 (dz)^2$$

on this neighborhood and this is holomorphic if $L(h)b^2$ is ; i.e. if

$$d(L(h)b^2) \equiv 0 \qquad \text{mod } dz \quad (\text{or } \varphi) \quad .$$

We compute

$$d\varphi = d(b\,dz) = db \wedge dz = \frac{db}{b} \wedge \varphi$$

$$d\varphi = d(\theta^1 + i\theta^2) = -\,\omega_2^1 \wedge \theta^2 - i\omega_1^2 \wedge \theta^1 = i\omega_2^1 \wedge \varphi \; .$$

Then

(3.2) $$db \equiv i\, b\, \omega_2^1 \quad \text{mod } dz \quad .$$

To compute $d(L(h))$ we begin with $\omega_i^2 = h_{ij}\,\theta^j$.

On M

$$\Omega_i^3 - \omega_k^3 \wedge \omega_i^k = d\omega_i^3 = dh_{ij} \wedge \theta^j - h_{ik}\,\omega_j^k \wedge \theta^j \quad \text{and}$$

$$-\omega_k^3 \wedge \omega_i^k = -\,h_{kj}\,\theta^j \wedge \omega_i^k$$

but $\Omega_i^3 = c\,\theta^3 \wedge \theta^i = 0$ on M . Then

(3.3) $$dh_{ij} - h_{ik}\,\omega_j^k - h_{jk}\,\omega_i^k = h_{ijk}\,\theta^k$$

where $$h_{ijk} = h_{jik} = h_{ikj} \qquad \forall\ i,j,k \quad .$$

We therefore obtain :

$$\begin{aligned} dL(h) &= \frac{1}{2}\,(dh_{11} - dh_{22}) - i\, dh_{12} \\ &= \frac{1}{2}\,(2h_{1k}\,\omega_1^k + h_{11k}\,\theta^k - 2h_{2k}\,\omega_2^k - h_{22k}\,\theta^k) - i(h_{1k}\omega_2^k + h_{2k}\omega_1^k + h_{12k}\theta^k) \\ &= 2h_{12}\omega_1^2 + \frac{1}{2}\,(h_{111} - h_{221})\theta^1 + \frac{1}{2}\,(h_{112} - h_{222})\theta^2 \ + \\ &\quad - i[(h_{22} - h_{11})\omega_1^2 + h_{121}\theta^1 + h_{122}\theta^2] \quad . \text{ Thus} \end{aligned}$$

(3.4) $$d\,L(h) = 2i\,L(h)\omega_1^2 + [\tfrac{1}{2}\,(h_{11k} - h_{22k}) - i\,h_{12k}]\,\theta^k \quad .$$

Suppose that $H = \frac{1}{2}\,(h_{11} + h_{22})$ is constant. Then from (3.3) we have :

$$0 = d(h_{11} + h_{22}) = 2(h_{12} - h_{21})\omega_1^2 + (h_{11k} + h_{22k})\theta^k \quad . \text{ Thus}$$

(3.5) $$h_{11k} + h_{22k} = 0 \qquad \forall\ k$$

and hence

$$\frac{1}{2}(h_{111} - h_{221}) - ih_{121} = - h_{221} - i\, h_{121} = - i(h_{112} - i\, h_{122}) \quad \text{and}$$

$$\frac{1}{2}(h_{112} - h_{222}) - ih_{122} = h_{112} - i\, h_{122} \quad ,$$

because of the symmetries of h_{ijk} (3,3) and (3.5) .
Therefore (3.4) becomes

$$d\, L(h) = 2i\, L(h)\omega_1^2 - i\, (h_{112} - i\, h_{122})\varphi \qquad \text{i.e.}$$

(3.6) $$d\, L(h) \equiv 2i\, L(h)\omega_1^2 \qquad \text{mod } dz \quad .$$

From (3.2) and (3.6) we deduce

$$\begin{aligned} d\,(b^2\, L(h)) &= 2b\, L(h)\, db + b^2\, d(L(h)) \\ &\equiv 2ib^2\, L(h)\omega_2^1 + 2ib^2\, L(h)\omega_1^2 \\ &\equiv 0 \quad \text{mod } z \end{aligned}$$

completing the proof of Lemma.

To get the proof of the Theorem, we use the Riemann-Roch Theorem which tells us that a quadric holomorphic form on S^2 must be identically zero.

4. A theorem of Chern's on a differential system.-

The following result is the essential tool needed to construct smooth higher order frame fields even on a neighborhood of an umbilical point (and their higher order analogues).

(4.1) Theorem (Chern [5]).

Let $w_\alpha(z)$ be complex valued functions which satisfy the differential system

$$\frac{\partial w_\alpha}{\partial \bar{z}} = \Sigma\, a_{\alpha\beta}\, w_\beta \quad , \quad 1 \leq \alpha, \beta \leq p \ ,$$

in a neighborhood of $z = 0$, where $a_{\alpha\beta}$ are complex valued C^1 functions. Suppose that the w_α do not all vanish identically. Then the w_α are of the form

$$w_\alpha(z) = z^m\, w^*_\alpha(z) \ ,$$

where m is an integer ≥ 0 and $w^*_\alpha(0)$ are not all zero.

5. Globally defined holomorphic forms and contact invariants.-

We now suppose that M is oriented and that M is isometrically minimally immersed in N , a space of constant sectional curvature c and dimension n . Let e be a local oriented Darboux frame (simply Darboux frame for short) defined on an open subset U of M . We saw in section 2 that two such frames e and $\tilde{e}$ are related on their common domain of definition $U \cap \tilde{U}$ by

$$(5.1) \qquad \tilde{e} = e\, K$$

where $K = \begin{pmatrix} A & o \\ o & B \end{pmatrix}$, $A : U \cap \tilde{U} \to SO(2)$, $B : U \cap \tilde{U} \to O(n-2)$.

We may express A in the form

$$(5.2) \qquad A = \begin{vmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{vmatrix} , \quad t : U \cap \tilde{U} \to \mathbb{R} \quad .$$

We define $S : U \to \mathbb{C}^{n-2}$ by

$$(5.3) \qquad S^\alpha = -h^\alpha_{11} + ih^\alpha_{12} = -L(h^\alpha) \qquad (i = \sqrt{-1}) \ .$$

By (2.19), a change of frame (5.1) transforms S by

$$(5.4) \qquad \tilde{S} = e^{2it} B^{-1} S \quad .$$

On the other hand, introducing on M the complex structure as in the proof of Hopf's theorem in § 3 by setting

$$(5.5) \qquad \varphi = \theta^1 + i\, \theta^2 \qquad i = \sqrt{-1}$$

we have, under a change of frame (5.1) on $U \cap \tilde{U}$,

$$(5.6) \qquad \tilde{\varphi} = e^{-it} \varphi \quad .$$

Hence (5.4) and (5.6) allow us to define globally on M a quartic symmetric form of bidegree (4,0) by setting

$$(5.7) \qquad \Phi = {}^t SS\, \varphi^4 \quad .$$

Proposition.- Φ is holomorphic.

Proof.

As in § 3 we set $\varphi = bdz$, where z is a local complex coordinate on some open set U and b is a smooth function never zero on U . Then Φ takes the form on U

$$(5.8) \qquad \Phi = b^4 \, \Sigma(S^\alpha)^2 \, dz^4 \quad .$$

To show that Φ is holomorphic, we have to show that

$$(5.9) \qquad d(b^4 \, \Sigma(S^\alpha)^2) \equiv 0 \quad \mathrm{mod}\ \varphi \quad (\text{or} \ \ dz) \ .$$

Upon taking the exterior derivative of (2.4) and using the structure equation (2.2), together with the fact that N has constant sectional curvature, we obtain

$$(5.10) \qquad dh^\alpha_{ik} - h^\alpha_{ij}\,\omega^j_k - h^\alpha_{jk}\,\omega^j_i + h^\beta_{ik}\,\omega^\alpha_\beta = a^\alpha_{ikm}\,\theta^m$$

with the smooth functions

$$(5.11) \qquad a^\alpha_{ikm} \quad \text{symmetric in } \ i,k,m \ \text{ for every } \ \alpha \ .$$

Taking the exterior derivative of the minimal surface equations $h^\alpha_{11} + h^\alpha_{22} = 0$ and using (5.10) , we get

$$(h^\beta_{11} + h^\beta_{22})\omega^\alpha_\beta = (a^\alpha_{11m} + a^\alpha_{22m})\,\theta^m \quad ,$$

and hence by minimality

$$(5.12) \qquad a^\alpha_{11m} + a^\alpha_{22m} = 0 \quad .$$

Using (3.2) , (5.3) and (5.10) , we calculate

$$(5.13) \quad d(b^4 \, \Sigma(S^\alpha)^2) \equiv$$

$$b^4 \sum_\alpha \{4i(S^\alpha)^2\omega^1_2 - 2\,S^\alpha[2h^\alpha_{1j}\,\omega^j_1 - h^\beta_{11}\omega^\alpha_\beta + a^\alpha_{11m}\,\theta^m - 2ih^\alpha_{11}\,\omega^1_2 + ih^\beta_{12}\,\omega^\alpha_\beta - ia^\alpha_{12m}\,\theta^m]\}$$

$$\equiv - 2b^4 \sum_\alpha S^\alpha \{S^\beta \omega_\beta^\alpha + (a_{11m}^\alpha - ia_{12m}^\alpha)\ \theta^m\}$$

$$\equiv - 2b^4 \sum_\alpha S^\alpha (a_{11m}^\alpha - ia_{12m}^\alpha)\ \theta^m \qquad \mathrm{mod}\ \varphi \quad .$$

But by (5.11) and (5.12)

$$(5.14) \qquad (a_{11m}^\alpha - ia_{12m}^\alpha)\ \theta^m = (a_{111}^\alpha - ia_{112}^\alpha)\ \varphi \quad .$$

Hence (5.14) and (5.13) give us (5.9) , which completes the proof.

From the Riemann-Roch theorem we have

Corollary.-

If M is the 2-sphere, then $\Phi = 0$.

Remarks.-

1. Even if M is not minimal in N we can define S by

$$(5.15) \qquad S^\alpha = \frac{1}{2}\ (h_{22}^\alpha - h_{11}^\alpha) + ih_{12}^\alpha = - L(h^\alpha) \ .$$

Then (5.4) still holds and Φ of (5.7) is still globally defined.

Proposition.-

If N is of constant sectional curvature c and the mean curvature vector H of M is parallel in the normal bundle ; i.e. $DH = 0$, then Φ is holomorphic.

2. By (5.4) the function

$$(5.16) \qquad \chi = {}^t S\bar{S}$$

is globally defined on M . Using (5.3) one obtains :

A point $p \in M$ is umbilic iff $\chi(p) = 0$.

3. In the case $n = 3$ we observe that $\chi(p) = 0$ iff $S(p) = 0$. Moreover (5.4) tells us that the quadratic symmetric form of type (1,0) given by

$$(5.17) \qquad \Gamma = S\,\varphi^2$$

is globally defined.

<u>Proposition</u>.-

Let N^3 be of constant curvature c . Suppose that the mean curvature vector H of M is parallel in the normal bundle and that M is topologically the 2-sphere. Then M is totally umbilic. If moreover M is minimal, then it is totally geodesic in N .

The following terminology was first introduced by Bryant [2] .

<u>Def</u>. The surface $M \to N$ is <u>superminimal</u> if the form Φ of (5.7) is identically zero.

We shall refer to this condition as <u>superminimal of order</u> 1.

The above Corollary shows that a minimal S^2 in N is always superminimal.

We suppose now superminimality : i.e.

$$(5.18) \qquad \Phi \equiv 0 \quad .$$

Hence, by (5.7), S is isotropic ; i.e. ${}^tSS = 0$. If we put

$$S = \mathrm{Re}\, S + i\, \mathrm{Im}\, S \quad \text{then}$$

$$(5.19) \qquad \langle \operatorname{Re} S , \operatorname{Im} S \rangle = 0 \quad ,$$

$$(5.20) \qquad |\operatorname{Re} S| = |\operatorname{Im} S| \quad ,$$

where $\langle\, ,\rangle$ and $|\cdot|$ denote the standard inner product and associated norm on $\mathbb{R}^{n-2}$.

From (5.3)

$$(5.21) \qquad |\operatorname{Re} S|^2 = \sum_\alpha (h^\alpha_{11})^2 \quad .$$

<u>Def</u>. The <u>second order</u> invariant of the superminimal surface M is

$$(5.22) \qquad r = |\operatorname{Re} S| \quad .$$

By (5.4), (5.19) and (5.20) r is globally defined on M , and by (5.21) r^2 is smooth.

If $r \equiv 0$ on M , then by (2.5) M is totally geodesic in N .

Assume now that r is <u>not</u> identically zero on M . Taking the exterior derivative of (5.3) and using (5.10), (5.11), (5.12), (5.5) and (5.3), we obtain

$$(5.23) \qquad dS^\alpha \equiv 2i\, S^\alpha \omega^2_1 - S^\beta\, \omega^\alpha_\beta \qquad \operatorname{mod}\ dz \quad .$$

Hence, using vector notation, we have

$$(5.24) \qquad \frac{\partial S}{\partial \bar z} = AS \quad ,$$

where A is an $(n-2) \times (n-2)$ matrix of smooth functions on U . Hence by Theorem 4.1 we have

Proposition.-

If M is superminimal and r is not identically zero, then the zeros of S are isolated in U, and consequently the zeros of r are isolated in M.

The zeros of r will be called the first order singular points of M. They are the umbilic points of M.

From (5.19), (5.20) and (5.22) it follows that a change of frame (5.1) can be made at any point of M such that $\tilde{S}$ of (5.4) takes the form

$$r(\varepsilon_1 + i\varepsilon_2) = r \begin{vmatrix} 1 \\ i \\ 0 \\ \vdots \\ 0 \end{vmatrix} , \qquad i = \sqrt{-1} \tag{5.25}$$

where $\varepsilon_1, \varepsilon_2$ are the first two vectors of the canonical basis of $\mathbb{R}^{n-2}$. This can be done smoothly on a neighborhood of any point where $r \neq 0$.

However, at a point p where $r(p) = 0$, we can bring S smoothly into the form

$$\lambda(\varepsilon_1 + i\varepsilon_2) , \tag{5.26}$$

where λ is a smooth complex valued function on a neighborhood of p such that $\lambda(p) = 0$ and

$$r = |\lambda| . \tag{5.27}$$

In fact, let z be a local complex coordinate about p such that $z(p) = 0$. Then by (4.1)

$$S = z^m S^* , \tag{5.28}$$

where $m \geq 0$ is an integer and $S^*(p) \neq 0$. But S^* is also isotropic, and a change of frame (5.1) will change S^* by (5.4) . Thus on a neighborhood of p one can smoothly transform S^* into

$$S^* = r^*(\varepsilon_1 + i\varepsilon_2) \quad , \tag{5.29}$$

where $r^* = |\mathrm{Re}\, S^*| > 0$. Thus by (5.28) S takes the form (5.26) where $\lambda = z^m r^*$.

Def. A local second order frame field along M is any local oriented first order frame field (i.e. any Darboux frame) with respect to which S of (5.3) takes the form (5.25) at each point.

A generalized local second order frame field along M is any Darboux frame with respect to which S takes the form (5.26) at each point.

Let $Z_1 = \{p \in M : r(p) = 0\}$, a discrete subset of M .

We summarize the above discussion as follows.

Proposition.-

A smooth generalized second order frame field exists on some neighborhood of any point of M .

A smooth second order frame field exists on some neighborhood of any point of $M \setminus Z_1$.

Let e be a smooth second order frame field on U . Then by (5.3) we have

$$\begin{aligned} \omega_1^3 &= - r\,\theta^1 \\ \omega_2^3 &= r\,\theta^2 \\ \omega_1^4 &= r\,\theta^2 \\ \omega_2^4 &= r\,\theta^1 \\ \omega_i^\gamma &= 0 \quad , \qquad \gamma \geq 5 \quad , \end{aligned} \tag{5.31}$$

where $r \geq 0$ is the second order invariant (5.22).

Let $\tilde{e}$ be a generalized smooth second order frame field on $\tilde{U}$. Then (5.3) says that

$$\begin{aligned}
\tilde{\omega}_1^3 &= -k\tilde{\theta}^1 + m\,\tilde{\theta}^2 \\
\tilde{\omega}_2^3 &= m\tilde{\theta}^1 + k\tilde{\theta}^2 \\
\tilde{\omega}_1^4 &= m\tilde{\theta}^1 + k\tilde{\theta}^2 \\
\tilde{\omega}_2^4 &= k\tilde{\theta}^1 - m\tilde{\theta}^2 \\
\tilde{\omega}_i^\gamma &= 0 \qquad\qquad , \qquad \gamma \geq 5 \ ,
\end{aligned} \tag{5.32}$$

where the smooth complex function λ on $\tilde{U}$ of (5.26) is written $\lambda = k + im$, where k and m are smooth real functions on $\tilde{U}$. Thus $\tilde{e}$ is second order iff $m \equiv 0$ on $\tilde{U}$. Notice that $r^2 = k^2 + m^2$.

If we take the exterior derivative of the equations $\tilde{\omega}_i^\gamma = 0$, and use the structure equations (2.2), (2.3), and then again (5.32), we obtain

$$\begin{aligned}
(k\,\tilde{\omega}_3^\gamma - m\,\tilde{\omega}_4^\gamma) \wedge \tilde{\theta}^1 - (m\,\tilde{\omega}_3^\gamma + k\,\tilde{\omega}_4^\gamma) \wedge \tilde{\theta}^2 &= 0 \\
(m\,\tilde{\omega}_3^\gamma + k\,\tilde{\omega}_4^\gamma) \wedge \tilde{\theta}^1 + (k\,\tilde{\omega}_3^\gamma - m\,\tilde{\omega}_4^\gamma) \wedge \tilde{\theta}^2 &= 0 \, .
\end{aligned} \tag{5.33}$$

Since the zeros of r are isolated, and all forms are smooth, it follows from Cartan's lemma that

$$\begin{aligned}
k\,\tilde{\omega}_3^\gamma - m\,\tilde{\omega}_4^\gamma &= \tilde{t}_{1j}^\gamma \ \tilde{\theta}^j \\
m\,\tilde{\omega}_3^\gamma + k\,\tilde{\omega}_4^\gamma &= \tilde{t}_{2j}^\gamma \ \tilde{\theta}^j
\end{aligned} \tag{5.34}$$

where

$$\tilde{t}_{12}^\gamma + \tilde{t}_{21}^\gamma = 0 \quad \text{and} \quad \tilde{t}_{11}^\gamma - \tilde{t}_{22}^\gamma = 0 \ , \tag{5.35}$$

and these are smooth functions on $\tilde{U}$.

Remark that for a second order frame field e on U the equations (5.32) specialize to (5.31), while (5.33) and (5.34) specialize to the case $k = r$, $m \equiv 0$.

We define smooth complex functions $\tilde{L}^\gamma$ on $\tilde{U}$ with respect to $\tilde{e}$ by

$$(5.36) \qquad \tilde{L}^\gamma = \tilde{t}^\gamma_{12} + i\,\tilde{t}^\gamma_{11} \qquad\qquad i = \sqrt{-1}$$

where $\tilde{t}^\gamma_{ij}$ are defined in (5.34) . We then put

$$\tilde{L} = (\tilde{L}^\gamma) : \tilde{U} \to \mathbb{C}^{n-4} \quad .$$

<u>Change of frame</u>. Suppose now that $U \cap \tilde{U} \neq \emptyset$ and contains no point of Z_1 . Then on this intersection, as in (5.1), $\tilde{e} = e\,K$ where

$$K = \begin{vmatrix} A & 0 & 0 \\ 0 & A^{-2}C & 0 \\ 0 & 0 & B \end{vmatrix} \quad ,$$

where A is given by (5.2) ,

$$C = \begin{vmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{vmatrix} \; : U \cap \tilde{U} \to SO(2) \quad \text{and} \quad B : U \cap \tilde{U} \to O(n-4) \; ,$$

are all smooth maps.

From (5.4), (5.25) and (5.26) we have

$$\lambda = e^{is}\, r \quad ,$$

and thus

$$(5.37) \qquad k = r\cos s \quad , \quad m = r\sin s \quad ,$$

since $\lambda = k + im$.

An application of (2.18) gives the following transformation law :

$$\tilde{\omega}^{\gamma}_{i+2} = (B^{-1})^{\gamma}_{\delta}\,\omega^{\delta}_{j+2}(A^{-2}C)^{j}_{i} \quad , \quad \gamma\ ,\ \delta > 5 \quad ; \quad i,j,k,m = 1,2 \ .$$

Hence by (5.34) and (5.37) and (5.6)

$$\tilde{t}^{\gamma}_{ij} = (B^{-1})^{\gamma}_{\delta}\, t^{\delta}_{km}(A^{-2})^{k}_{i}\, A^{m}_{j} \quad , \tag{5.38}$$

from which we conclude (using (5.35) and (5.36)) that L transforms by

$$\tilde{L} = e^{3it}\, B^{-1}\, L \quad . \tag{5.39}$$

It follows from (5.6), (5.36) and (5.39) that the smooth symmetric bidegree (6,0) form

$$\Lambda = {}^{t}LL\ \varphi^{6} \tag{5.40}$$

is globally defined on M. The superminimality of M in N implies :

<u>Proposition</u>.-

Λ is holomorphic.

<u>Proof</u>.

Set $\varphi = h\,dz$ on U. Then Λ on U with respect to the second order frame e is given by

$$\Lambda = h^{6} \sum_{\gamma} (L^{\gamma})^{2}\, dz^{6} \quad . \tag{5.41}$$

We need to show

$$d(h^{6}\,\Sigma(L^{\gamma})^{2}) \equiv 0 \quad \text{modulo } \varphi \quad (\text{or } dz) \quad , \tag{5.42}$$

on U. This will prove that Λ is holomorphic on $M \setminus Z_1$. But then since Λ

is smooth on M , the points of Z_1 must be removable singularities of Λ .

Taking the exterior derivative of (5.34), (where now $k = r$ and $m = 0$) using the structure equation (2.2) and then (5.34) again, and using the fact that N has constant sectional curvature, we have

$$(5.43) \quad \left(-T^{\gamma}_{jk} \frac{dr}{r} + dt^{\gamma}_{jk} - t^{\gamma}_{j} \ \omega^{i}_{k} - t^{\gamma}_{ik} \ \omega^{i+2}_{j+2} + t^{\delta}_{jk} \ \omega^{\gamma}_{\delta}\right) \wedge \theta^{k} = 0 \quad , \quad \delta, \gamma \geqslant 5 .$$

Thus by Cartan's lemma

$$(5.44) \quad -T^{\gamma}_{jk} \frac{dr}{r} + dt^{\gamma}_{jk} - t^{\gamma}_{j1} \ \omega^{i}_{k} - t^{\gamma}_{ik} \ \omega^{i+2}_{j+2} + t^{\delta}_{jk} \ \omega^{\gamma}_{\delta} = b^{\gamma}_{jkm} \ \theta^{m} \quad ,$$

with

$$(5.45) \qquad b^{\gamma}_{jkm} = b^{\gamma}_{jmk} \qquad \delta, \gamma \geqslant 5 .$$

Furthermore, exterior derivative of (5.35) gives

$$(5.46) \qquad b^{\gamma}_{121} = -b^{\gamma}_{211} \quad , \quad b^{\gamma}_{122} = -b^{\gamma}_{212} \quad ; \text{ and}$$

$$(5.47) \qquad b^{\gamma}_{111} = b^{\gamma}_{221} \quad , \quad b^{\gamma}_{112} = b^{\gamma}_{222} \quad .$$

We need some information about dr . For this we take the exterior derivative of the first four equations in (5.31). Using the structure equations (2.1) and (2.2), and again (5.31), and the fact that N has constant sectional curvature we have :

From $\omega^{4}_{1} = r \ \theta^{2}$:

$$(5.48) \qquad -r(\omega^{4}_{3} - 2\omega^{1}_{2}) \wedge \theta^{1} + dr \wedge \theta^{2} = 0$$

and from $\omega^{4}_{2} = r \ \theta^{1}$

$$(5.49) \qquad dr \wedge \theta^{1} + r(\omega^{4}_{3} - 2\omega^{1}_{2}) \wedge \theta^{2} = 0 \quad .$$

Then from Cartan's lemma we have

(5.50) $$dr = a\theta^1 + b\theta^2$$

$$r(2\omega_2^1 - \omega_3^4) = -b\theta^1 + a\theta^2 \qquad a,b \ \text{smooth functions on} \ U \ .$$

Remark : From the first two equations of (5.31) we again obtain (5.50).

Hence

(5.51) $$dr + ir(2\omega_2^1 - \omega_3^4) \equiv 0 \quad \text{mod } \varphi \ ,$$

and from (5.47), (5.46) and (5.45)

(5.52) $$(b^{\gamma}_{12m} + ib^{\gamma}_{11m})\ \theta^m \equiv 0 \quad \text{mod } \varphi \ .$$

With this preparation (5.42) becomes a routine calculation.

Remarks.

1. The existence of the holomorphic from Λ requires that M be superminimal, which allows us to construct (generalized) second order frame fields.

2. If M is the 2-sphere, then $\Lambda \equiv 0$.

Def. The surface $M \to N$ is said to be superminimal of order 2 if it is superminimal and Λ of (5.40) is identically zero.

Suppose now that $M \to N$ is superminimal of order 2. Then with respect to any generalized second order frame field the complex vector $L : U \to \mathbb{C}^{n-4}$ of (5.36) is isotropic. Proceeding as in the first order case, we set

(5.54) $$\mathscr{L} = |\mathrm{Re}\ L| = |\mathrm{Im}\ L| = \frac{1}{\sqrt{2}}\ |L| \ .$$

It follows from (5.39) that $\mathscr{A}^2 = |L|^2$ is a globally defined smooth function on M. We shall call $\mathscr{A}$ the <u>third order</u> invariant of M in N.

As we did for r, we can show that either $\mathscr{A} \equiv 0$ or $\mathscr{A}$ has isolated zeros. We shall consider the former case later. Suppose that $\mathscr{A} \not\equiv 0$, and let Z_2 denote its discrete set of zeros. By (5.39) at each point of M there exists a second order frame with respect to which

$$(*) \qquad L = \mathscr{A} \begin{vmatrix} 1 \\ i \\ 0 \\ \vdots \\ 0 \end{vmatrix} \in \mathbb{C}^{n-4} \qquad i = \sqrt{-1} \; .$$

<u>Def</u>. A local third order frame field on M is a local second order frame field which, in addition to (5.31), satisfies

$$(5.55) \qquad \begin{aligned} \omega_3^5 &= s\,\theta^2 \\ \omega_4^5 &= -s\,\theta^1 \\ \omega_3^6 &= s\,\theta^1 \\ \omega_4^6 &= s\,\theta^2 \\ \omega_{i+2}^{\mu} &= 0 \qquad , \quad \mu \geqslant 7 \end{aligned} \qquad\qquad s = \frac{\mathscr{A}}{r}$$

Smooth local third order frame fields exist on some neighborhood of any point in $M - \{Z_1 \cup Z_2\}$. If we define <u>generalized</u> local third order frame fields to be generalized local second order frame fields which also satisfy $(*)$ with $\mathscr{A}$ replaced by a complex function $\tilde{\mathscr{A}}$, then smooth generalized local third order frame fields exist on some neighborhood of any point of M. For such frames $|\tilde{\mathscr{A}}| = \mathscr{A}$.

The procedure should now be clear for the construction of the higher order invariants $r, s, \ldots$ and the higher order frames on surfaces of sufficiently high order of superminimality (which is always satisfied by minimal spheres).

To facilitate the ease of exposition we shall confine ourselves to the case $n \leqslant 6$.

6. Calabi's Theorem.-

The general preparations of the last section give us the means for a simple proof of the following result which was first proved by E. Calabi [3] for the case of minimal spheres in S^n .

Theorem.-

Let $f : M \to N^n$ be a connected superminimal immersion of order $[\frac{n}{2}]$ which is not contained in any totally geodesic hypersurface of N . Then n is even.

Proof.

We consider the case $n = 5$. The general case is proved in the same way. By hypothesis now f is superminimal of order 2.

If the second order invariant r of (5.22) is identically zero, then f is totally geodesic. Thus we may assume that $r \not\equiv 0$, and therefore Λ of (5.40) vanishing says that $L^2 = {}^tLL \equiv 0$ on U .

Hence $L = 0$, which by (5.38) says that $t^5_{12} = 0 = t^5_{11}$. It follows from (5.35) and (5.34) that

$$\omega^5_3 = \omega^5_4 = 0 \quad , \tag{6.1}$$

for any generalized second order frame field on M .

Of course for any such frame field we also have

$$\omega^5_1 = \omega^5_2 = \theta^5 = 0 \quad . \tag{6.2}$$

If we interpret θ^A and ω^A_B , $1 \leqslant A,B \leqslant 5$, as the canonical forms and connection forms on the bundle of orthonormal frames $O(N)$ of N , then (6.1) and (6.2) define smooth distribution on $O(N)$ which satisfies the Frobenius condition since N has constant sectional curvature. It follows from (6.1) and (6.2) that the bundle of generalized second order frames on M is an integral submanifold of this distribution on $O(N)$, and hence is contained in a connected maximal integral submanifold O^1 . Thus $f(M)$ is contained in the projection $\pi(O^1)$, which is a totally geodesic hypersurface of N , where $\pi : O(N) \to N$ is the projection.

7. Veronese surfaces.-

Suppose that $f : M \to N^6$ is superminimal of second order. Let r and s denote the second and third order invariants defined in (5.22) and (5.55), respectively. Thus we are assuming that $r \not\equiv 0$, and hence its set of zeros Z_1 is discrete. Let e be a third order frame field on $U \subseteq M \smallsetminus (Z_1 \cup Z_2)$.

From (5.31), taking exterior derivative of ω^1_2 , and using (2.2) and (2.3), we obtain

$$(7.1) \qquad K = c - 2r^2 ,$$

where K is the Gaussian curvature of M , c is the constant sectional curvature of N .

Remark. Our calculations have established (7.1) on $M \smallsetminus (Z_1 \cup Z_2)$. By continuity it will hold on M . This reasoning is used repeatedly in this section.

Recall equations (5.50). We derive analogous equations for s as follows.

Take the exterior derivative of the first two equations in (5.55), use (2.2), (2.3) and (5.55) again plus Cartan's lemma, to get

$$(7.2) \qquad ds = \tilde{a}\,\theta^1 + \tilde{b}\,\theta^2$$

$$s(\omega_5^6 - \omega_3^4 - \omega_2^1) = \tilde{b}\,\theta^1 - \tilde{a}\,\theta^2$$

where $\tilde{a}$, $\tilde{b}$ are smooth functions on U . (Nothing else is obtained from the third and fourth equations of (5.55)).

<u>Constant Gaussian curvature</u> K .-

Suppose that K is constant. From (7.1) r is constant, hence positive, and (5.50) gives

$$(7.3) \qquad 2\,\omega_2^1 = \omega_3^4 \quad .$$

Taking the exterior derivative of (7.3) and using (7.1) and (7.3), we get

$$(7.4) \qquad K = r^2 - s^2 \quad .$$

Hence s is also constant.

If $s = 0$, then by (7.4) $K = \frac{c}{3}$.

If $s > 0$, then from (7.2)

$$(7.5) \qquad \omega_5^6 - \omega_3^4 = \omega_2^1 \quad .$$

Taking the exterior differential of (7.5) and using (2.2), (2.3), (5.31) and (5.55), we get

$$(7.6) \qquad K = 4\,s^2 - 2\,r^2 \quad ,$$

when K is constant and s is positive (necessarily constant).
Hence, under these hypotheses, (7.1), (7.4) and (7.6) combine to give

$$(7.8) \qquad K = \frac{c}{6} \quad .$$

Up to an isometry of N these is only one surface satisfying (7.8) . In the case of $N = S^{2p}$ it is the Veronese surface.

In the case $n = 4$ and K constant we must stop at equation (7.4) which then becomes $K = r^2$, which combines with (7.1) to give $K = \frac{c}{3}$.

In general, for minimal $S^2 \to S^{2p}$ not contained in a totally geodesic S^{2p-2} , which has constant Gaussian curvature K , we have

$$K = \frac{2}{p(p+1)} \quad .$$

This was first proved by Calabi [3] .

The case of vanishing s .-

We return to the general case, with $r \neq 0$, but now we suppose that $s \equiv 0$. It follows from (5.54) that $L \equiv 0$ with respect to any generalized second order frame field, and hence any such frame field is also third order by (5.36), (5.34) and (5.55) .

Consequently, any such frame field is an integral submanifold of the distribution defined on the bundle of orthonormal frames $O(N)$ by the equations

$$(7.9) \qquad \omega_1^5 = \omega_2^5 = \omega_3^5 = \omega_4^5 = \omega_1^6 = \omega_2^6 = \omega_3^6 = \omega_4^6 = \theta^5 = \theta^6 = 0 \quad .$$

This distribution is completely integrable, since N has constant curvature, and any of its maximal connected integral submanifolds projects by $\pi : O(N) \to N$ onto a totally geodesic submanifold of N of codimension 2.

Proposition.-

If superminimal immersion $f : M \to N^6$ has $r \not\equiv 0$, but $s \equiv 0$, then $f(M)$ is contained in a totally geodesic N^4 .

8. The Frenet-Boruvka formulas and the directrix curve.-

Consider now the simplest case : S^2 minimally immersed in S^4 , but not totally geodesically. Let $e = (e_1, \dots, e_4)$ be a local Darboux frame field along $f : S^2 \to S^4$. We interpret the e_A as $\mathbb{R}^5$-valued functions defined on $U \subset S^2$, and define $\mathbb{C}^5$-valued functions

(8.1) $$E_1 = e_1 + ie_2 \quad , \quad E_2 = e_3 + ie_4 \quad .$$

A change of Darboux frame field (5.1) will transform the E_i by

(8.2) $$\tilde{E}_1 = e^{it} E_1 \quad , \quad \tilde{E}_2 = e^{i\tau} E_2 \quad ,$$

where t and τ are smooth functions on $U \cap \tilde{U}$.
It follows that the smooth maps

(8.3) $$[E_i] \ : \ S^2 \to \mathbb{C}P^4 \quad , \quad i = 1,2$$

are globally defined, where $[\cdot]$ denotes the equivalence class in $\mathbb{C}P^4$ of a non-zero vector in $\mathbb{C}^5$.

If $(\, , \,)$ denotes the symmetric scalar product in $\mathbb{C}^5$, then from (8.1) we have

(8.4) $$(E_1, E_1) = (E_1, \bar{E}_2) = (E_2, E_2) = 0 \, ,$$

and the complex conjugates of (8.4) also hold.

From equation (2.20) we have

$$\text{(8.5)} \qquad df = \frac{1}{2} E_1 \bar{\varphi} + \frac{1}{2} \bar{E}_1 \varphi \quad .$$

Suppose now that e is a second order frame field. Then from equations (2.20) and (5.31) we have

$$\text{(8.6)} \qquad \begin{aligned} dE_1 &= - f \varphi + i E_1 \omega^1_2 - r \bar{E}_2 \bar{\varphi} \\ dE_2 &= i E_2 \omega^3_4 + r \bar{E}_1 \bar{\varphi} \quad . \end{aligned}$$

Chern [4] calls (8.5) and (8.6) the Frenet-Boruvka formulas for f. The second equation of (8.6) shows that the map

$$\text{(8.7)} \qquad F = [\bar{E}_2] : S^2 \to \mathbb{C}P^4$$

is holomorphic. The second order invariant r is the length of its tangent vector, or in other words, the pull-back by F of the Fubini-Study metric of $\mathbb{C}P^4$ is $r^2 \varphi \bar{\varphi}$. Equations (8.4) show that F actually takes values in the non-singular quadric in $\mathbb{C}P^4$.

$$Q_3 = \{[Z] \in \mathbb{C}P^4 \ : \ {}^tZZ = 0\} \quad .$$

Moreover its tangent line at each point, the span of E_1 and $\bar{E}_2$, is also in Q_3 .

Chern [4] calls F the directrix curve of the minimal surface f. He has shown that if one begins with a holomorphic isotropic curve $F : S^2 \to Q_3$, then one can construct a minimal surface $f : S^2 \to S^4$ whose directrix curve is F .

9. Geometric interpretation of the invariants.-

Let M be a superminimal surface of first order in the space N of constant curvature c. Let $TM^{\perp}$ denote the normal bundle of M in N. If we use a local second order frame field then equations (5.31) together with (2.4) and (2.17) give

$$(9.1) \qquad {}^{\perp}R_{3412} = -2r^2$$

$$(9.2) \qquad {}^{\perp}R_{\gamma\delta ij} = 0 \qquad \text{for } \gamma, \delta \geq 5 .$$

Combined with (7.1) these equations give

Proposition.-

The Gaussian curvature K of M is

$$(9.3) \qquad K = c + {}^{\perp}R_{4312} ,$$

where ${}^{\perp}R_{\alpha\beta ij}$ are the components of the normal curvature tensor (cf. 2.15).

Suppose now that M is superminimal of second order in N (and thus $r \neq 0$). If $e = (e_1, \ldots, e_n)$ is a generalized second order frame field, then the 2-plane spanned at each point by e_3, e_4 does not depend on the choice of e. Hence they define a C^{∞} rank 2 vector bundle P on M, a subbundle of $TM^{\perp}$.

If we set

$$(9.4) \qquad {}^{P}D : \Gamma(P) \to \Gamma(P \otimes T^{*}M) ,$$

$${}^{P}D\, e_{\alpha} = e_{\beta}\, \omega_{\alpha}^{\beta} \qquad \alpha, \beta = 3,4$$

(where $e = (e_1, \dots, e_n)$ is any generalized second order frame field then ${}^P D$ is a covariant derivation in P .

To compute the curvature of ${}^P D$ we use a third order frame field e . Then by (5.31) and (5.55) we obtain

$$d\,\omega_4^3 = 2(s^2 - r^2)\,\theta^1 \wedge \theta^2 \tag{9.5}$$

and hence

$${}^P K = 2(s^2 - r^2) \quad . \tag{9.6}$$

Remark : If $n = \dim N = 4$, then $P = TM^{\perp}$ and ${}^P K = {}^{\perp}K = -2r^2$.

Suppose now that $n \geq 6$. Then we can consider the C^∞ vector bundle Q over M admitting as a local frame field e_γ , $\gamma \geq 5$, where e is a generalized second order frame field on M . We introduce a covariant derivative in Q

$$\begin{aligned} &{}^Q D : \Gamma(Q) \to \Gamma(Q \otimes T^*M) \quad , \\ &{}^Q D\, e_\gamma = e_\delta\, \omega_\gamma^\delta \qquad , \qquad \gamma, \delta \geq 5 \quad . \end{aligned} \tag{9.7}$$

Proceeding as in § 2, using (5.34) and the fact that N has constant sectional cruvature, one obtains

$$r^2 \ {}^Q R_{\gamma\delta jk} = t_{ij}^\gamma\, t_{ik}^\delta - t_{ik}^\gamma\, t_{ij}^\delta \qquad \text{(sum on } i) , \tag{9.8}$$

where ${}^Q R_{\gamma\delta jk}$ are the components of the curvature tensor of Q and t_{ij}^γ are the funtions in (5.34) .

If we specialize further to third order frames e , then from (5.55), (5.34) and (9.8) we obtain

$${}^Q R_{5612} = -2s^2 \quad , \quad {}^Q R_{\gamma\delta jk} = 0 \quad \text{if } \gamma \text{ or } \delta \text{ is} \geq 7 \ . \tag{9.9}$$

Equations (7.1), (9.6), (9.9) combined prove the following :

Proposition.-

Let M be a superminimal of second order surface in a space N of constant sectional curvature c . Then the Gaussian curvature K of M is given by :

$$K = c + {}^{P}K + {}^{Q}R_{5612} \tag{9.10}$$

where ${}^{P}K$ is the curvature of the 2-plane bundle P and ${}^{Q}R_{\gamma\delta ij}$ are the components of the curvature tensor in the bundle Q .

We observe some immediate consequences of these two propositions in the cases $n = 4, 6$. In what follows M is assumed to be compact.

Case $n = 4$.-

In this case we can rewrite (9.3) as

$$K = c + {}^{P}K \tag{9.11}$$

where ${}^{P}K = -2r^2$ is the curvature in $TM^{\perp} = P$. Integrating both rides of (9.11) over M and using the Chern-Gauss-Bonnet Theorem we obtain :

$$2\pi(\chi(M) - \chi(P)) = c\,A \tag{9.12}$$

where $\chi(M)$, $\chi(P)$ are the Euler chcaracteristics respectively of M and of the bundle P , and A is the area of the superminimal surface $M \to N$.

We remark also that since ${}^{P}K = -2r^2$, then :

In particular let $S^2 \to S^4$ minimally then (9.12) becomes

$$A = 2\pi(2 - \chi(P)) \quad . \tag{9.14}$$

Since $\chi(P)$ is always on even number (it is twice the self intersection number of the immersion), it follows that A is a multiple of 4π (first proved by Barbosa [1]) . Moreover using ${}^{P}K = -2r^2$, (9.1), (9.2), (9.13) we have that the following are equivalent :

$$(9.15) \qquad \begin{array}{rl} \text{i)} & S^2 \text{ is totally geodesic in } S^4 \\ \text{ii)} & r \equiv 0 \\ \text{iii)} & \chi(P) = 0 \\ \text{iv)} & {}^{\perp}R \equiv 0 \qquad \text{i.e. } P \text{ is flat .} \end{array}$$

Consequently the following are equivalent :

$$(9.16) \qquad \begin{array}{rl} \text{j)} & S^2 \text{ is not totally geodesic in } S^2 \\ \text{jj)} & \chi(P) < 0 \quad . \end{array}$$

Remarks : 1. Equivalence of i) and iii) was first proved by Ruh [15] .

2. In the case where $S^2 \to S^4$ is the Veronese surface, i.e. $K \equiv \frac{1}{3}$, we have $A = 12\pi$ and hence by (9.14)

$$(9.17) \qquad \chi(P) = -4 \quad .$$

Case $n = 6$.-

In this case we rewrite (9.10) as

$$(9.18) \qquad K = c + {}^{P}K + {}^{Q}K$$

and again integrating as before we get :

$$(9.19) \qquad 2\pi(\chi(M) - \chi(P) - \chi(Q)) = c\,A \ .$$

In this case, since ${}^{Q}K = -2s^2$, we obtain

$$\chi(Q) \leq 0 \quad . \tag{9.20}$$

In particular for $S^2 \to S^6$ minimally

$$A = 2\pi(2 - \chi(P) - \chi(Q)) \quad . \tag{9.21}$$

Moreover using ${}^{Q}K = -2s^2$ and (9.20) we obtain the equivalence of the following :

$$\begin{array}{rl} \text{i)} & s \equiv 0 \\ \text{ii)} & \chi(Q) = 0 \\ \text{iii)} & {}^{Q}K \equiv 0 \quad \text{i.e. } Q \text{ is flat} \end{array} \tag{9.22}$$

and the equivalence of

$$\begin{array}{rl} \text{j)} & s \not\equiv 0 \\ \text{jj)} & \chi(Q) < 0 \quad . \end{array} \tag{9.23}$$

Using the geometric interpretation of $s \equiv 0$ given in § 7, and the meaning of the coframe and connection forms in this case given in § 2, equivalence of i) and ii) in (9.22) implies the following :

<u>Proposition</u>.-

Let S^2 be minimally immersed in S^6, not totally geodesically. Then S^2 is contained in S^4 totally geodesic in S^6 iff $\chi(Q) = 0$, where Q is the bundle locally spanned by e_5, e_6 for $\{e_i\}$ a generalized second order frame.

<u>Remark</u> : In the case $S^2 \to S^6$ as a Veronese surface, i.e. K constant, we have the following two possibilities :

i) $K \equiv \frac{1}{3}$ i.e. (9.22) in which case $A = 12\pi$, $\chi(Q) = 0$ and from (9.21) we have $\chi(P) = -4$.

ii) $K \equiv \frac{1}{6}$ i.e. (9.23). We can then compute that $A = 24\pi$. Moreover from (7.7) and (7.6) we have :

$$s^2 = \frac{1}{4} \quad . \tag{9.24}$$

Then from (9.24), the previous estimate of A and Chern-Gauss-Bonnet Theorem we have :

$$\chi(Q) = -6 \ . \tag{9.25}$$

Again (9.21) gives us $\chi(P) = -4$.

Rigidity.-

Our general constructions above give the means for a simple proof of the following result of Barbosa [1] .

Theorem.-

Let $f,g : S^2 \to S^{2p}$ be two minimal immersions not lying in any totally geodesic S^{2q} , $q < p$. Then f and g are isometric iff they differ by a rigid motion of S^{2p} .

Proof.

We sketch a proof for the case $p = 3$. If f and g are isometric then they have the same Gaussian curvature, and hence the same second order invariant, by (7.1). But then their third order invariants must be equal since $2K + {}^{P}K = \Delta \log r$ (see Rigoli [14], or Chern [5]) and by (9.6). Hence, with the same r and s , it follows from (5.31) and (5.55) that the pull-backs are equal of the Maurer-Cartan form of $O(7) = O(S^6)$ by any pair of third order frame fields along f and g , respectively. The theorem then follows from the rigidity theorem of Cartan-Darboux.

Quantization.-

By the techniques of decomposing the normal bundle into a sum of rank two subbundles and analysing the laplacian of the logarithm of the invariants, Rigoli [14] has proved the following result.

Theorem.-

Suppose $f : S^2 \to S^{2p}$ is minimal and not contained in any totally geodesic S^{2p-2}. If the Gaussian curvature

$$K \geq \frac{2}{p(p+1)} ,$$

then $K = \dfrac{2}{p(p+1)}$, and f is the Veronese surface.

§ 10.- Quaternionic projective space.

The quaternions $\mathbb{H} = \{z + jw : z,w \in C\}$, where j is a quaternion satisfying $j \cdot j = -1$ and $jw = \bar{w}j$, for every $w \in \mathbb{C}$. $\mathbb{H}$ is a right $\mathbb{C}$-module. It possesses a conjugation given by

$$\overline{z + jw} = \bar{z} - jw .$$

Then $\overline{pq} = \bar{q}\bar{p}$ for any $p,q \in \mathbb{H}$.

$\mathbb{H}^n$ denotes the set of all n-tuples of quaternions, written as columns. It is both a right $\mathbb{H}$-module and a right $\mathbb{C}$-module. There is a canonical isomorphism $\mathbb{H}^n \cong \mathbb{C}^{2n}$ given by

(10.1) $$z + jw \leftrightarrow \begin{pmatrix} z \\ w \end{pmatrix} \text{ , where } z,w \in \mathbb{C}^n .$$

We define an $\mathbb{H}$-valued bilinear form on $\mathbb{H}^n$ by

$$(p,q) = {}^t\bar{p}q .$$

Then for $\lambda \in \mathbb{H}$, $(p\lambda, q) = \bar{\lambda}(p,q)$, and $(p,q\lambda) = (p,q)\lambda$, and $\overline{(p,q)} = (q,p)$. This bilinear form is positive definite in the sense that $(p,p) > 0$ if $p \neq 0 \in \mathbb{H}^n$. The real part of (p,q) , namely $\mathrm{Re}(p,q) = \frac{1}{2}((p,q) + \overline{(p,q)})$, is the standard inner product on $\mathbb{R}^{4n} \cong \mathbb{H}^n$.

Quaternionic projective space $\mathbb{H}P^n$ is the space of one dimensional $\mathbb{H}$-subspaces of $\mathbb{H}^{n+1}$. Thus $\mathbb{H}P^n = \mathbb{H}^{n+1} \setminus \{0\}/\mathbb{H}^*$, where $\mathbb{H}^* = \mathbb{H} \setminus \{0\}$. We let $\rho : \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{H}P^n$ denote the canonical projection $\rho(p) = [p] = p \cdot \mathbb{H}$.

$\mathbb{H}P^n$ is a real analytic simply connected manifold of (real) dimension 4n acted on transitively by the simplectic group

$$Sp(n+1) = \{u \in GL(n+1;\mathbb{H}) : {}^t\bar{u}\, u = I\} .$$

For any $u \in Sp(n+1)$ we write $u = (e_0, e_1, \ldots, e_n)$, where $e_A \in \mathbb{H}^{n+1}$ is the A^{th} column of u , and we establish the index convention $0 \leq A,B,C \leq n$. The projection $\pi : Sp(n+1) \to \mathbb{H}P^n$ is defined by $\pi(u) = u \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = [e_0]$.

$Sp(n+1)$ is a compact real Lie group of dimension $2n^2 + 5n + 3$. For small n it is closely related to the sphere, namely $Sp(1) \cong Spin(3)$ and $Sp(2) \cong Spin(5)$, the simply connected double covers of $SO(3)$ and $SO(5)$ respectively.

The Lie algebra of $Sp(n+1)$ is

$$sp(n+1) = \{X \in \mathfrak{gl}(n+1; \mathbb{H}) : {}^t\bar{X} + X = 0\} .$$

The Maurer-Cartan form Φ of $Sp(n+1)$, written as a left-invariant 1-form with values in $sp(n+1)$, is given by

$$(10.2) \qquad \Phi_{(u)} = u^{-1}\, du = {}^t\bar{u}\, du \quad .$$

The Maurer-Cartan structure equations are

$$d\Phi = - \Phi \wedge \Phi \quad .$$

If we write $\Phi = (\Phi^A_B)$, where Φ^A_B is a left-invariant 1-form on $SP(n+1)$ with values in $\mathbb{H}$, then

$$\Phi^A_B = - \bar{\Phi}^B_A \quad ,$$

and (10.2) becomes

$$(10.3) \qquad de_A = e_B\, \Phi^B_A \quad ,$$

while the structure equations become

$$d\Phi^A_B = - \Phi^A_C \wedge \Phi^C_B \quad .$$

The $\mathbb{H}$-valued forms Φ^A_B can be decomposed into

$$\Phi^A_B = \alpha^A_B + j\,\beta^A_B \,, \tag{10.4}$$

where α^A_B and β^A_B are left-invariant $\mathbb{C}$-valued 1-forms on $Sp(n+1)$ which satisfy

$$\bar{\alpha}^A_B = -\alpha^B_A \,, \quad \beta^A_B = \beta^B_A \,. \tag{10.5}$$

Then the structure equations become

$$\begin{aligned} d\alpha^A_B &= -\alpha^A_C \wedge \alpha^C_B + \bar{\beta}^A_C \wedge \beta^C_B \\ d\beta^A_B &= -\bar{\alpha}^A_C \wedge \beta^C_B - \beta^A_C \wedge \alpha^C_B \end{aligned} \,. \tag{10.6}$$

The isotropy subgroup of $Sp(n+1)$ at the point $0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ of $\mathbb{H}P^n$ is

$$G_0 = \left\{ \begin{vmatrix} a & o \\ o & A \end{vmatrix} : a \in Sp(1) \,, \; A \in Sp(n) \right\} \cong Sp(1) \times Sp(n) \,.$$

Its Lie algebra is

$$\mathfrak{g}_0 = \left\{ \begin{vmatrix} x & o \\ o & X \end{vmatrix} : x \in sp(1) \,, \; X \in sp(n) \right\} = sp(1) \times sp(n) \,,$$

and an $Ad(G_0)$-invariant complementary subspace to $\mathfrak{g}_0$ in $sp(n+1)$ is given by

$$\mathfrak{m}_0 = \begin{vmatrix} o & -{}^t\bar{p} \\ p & o \end{vmatrix} : p \in \mathbb{H}^n \,.$$

The decomposition $sp(n+1) = \mathfrak{g}_0 + \mathfrak{m}_0$ decomposes Φ into $\Phi = \Phi_{\mathfrak{g}_0} + \Phi_0$ where

$$\Phi_0 = \begin{vmatrix} o & \Phi^0_a \\ \Phi^a_0 & o \end{vmatrix}$$, where we establish another index convention $1 \le a,b,c \le n$.

Up to constant positive factor there exists a unique $Sp(n+1)$-invariant Riemannian metric g on $\mathbb{H}P^n$. It is determined by the $Ad(G_o)$-invariant inner product on $\mathfrak{m}_o$ given by

$$(10.7) \qquad \langle\, , \rangle = \sum_1^n \bar{\Phi}_o^a \, \Phi_o^a \quad , \quad \text{(symmetric product)} .$$

In fact the adjoint action of G_o on $\mathfrak{m}_o$ can be written as follows. We have a canonical isomorphism

$$\mathfrak{m}_o \cong \mathbb{H}^n \text{ by } \begin{vmatrix} o & -{}^t\bar{p} \\ p & o \end{vmatrix} \longleftrightarrow p \text{ . Then for } K = \begin{vmatrix} a & o \\ o & A \end{vmatrix} \in G_o \; ,$$

$$(10.8) \qquad Ad(K)p = Apa^{-1} \; ,$$

which clearly preserves the inner product $\langle\, , \rangle$.

Def. A <u>local</u> <u>symplectic</u> <u>frame</u> <u>field</u> in $\mathbb{H}P^n$ is a local section $u : U \to Sp(n+1)$ of the principal G_o-bundle $\pi : Sp(n+1) \to \mathbb{H}P^n$, where U is an open subset of $\mathbb{H}P^n$.

Then $\{\varphi^a = u^* \Phi_o^a\}$ is an $\mathbb{H}$-valued coframe field on U . In fact, writing $\varphi^a = \alpha_o^a + j\, \beta_o^a$ as in (10.4), and $\alpha_o^a = \theta^a + i\, \theta^{n+a}$, $\beta_o^a = \psi^a + i\, \psi^{n+a}$, then $\{\theta^a, \theta^{n+a}, \psi^a, \psi^{n+a}\}$ is a real coframe field on U, which is moreover orthonormal ; i.e.

$$(10.9) \qquad g = \sum_1^n (\theta^a)^2 + (\theta^{n+a})^2 + (\psi^a)^2 + (\psi^{n+a})^2 \; .$$

We consider the real and imaginary parts of the complex valued forms in (10.4) :

$$\alpha^o_o = i\ \rho_o \quad , \quad \alpha^a_b = \omega^a_b + i\ \rho^a_b$$

(10.10)

$$\beta^o_o = \mu_o + i\ \nu_o \quad , \quad \beta^a_b = \mu^a_b + i\ \nu^a_b$$

where all 1-forms on the right side of the equations are real valued, and by (10.5)

$$(10.11) \qquad \omega^a_b = -\ \omega^b_a \quad , \quad \rho^a_b = \rho^b_a \ .$$

Using the structure equations (10.6) one can obtain the Levi-Civita connections forms and the curvature forms of g with respect to the orthonormal coframe field in (10.8) . In the case $n = 1$ we get

$$(10.12) \qquad d\begin{vmatrix} \theta^1 \\ \theta^2 \\ \psi^1 \\ \psi^2 \end{vmatrix} = -\begin{vmatrix} 0 & -\rho^1_1 + \rho_o & -\mu^1_1 + \mu_o & -\nu^1_1 + \nu_o \\ \rho^1_1 - \rho_o & 0 & \nu^1_1 + \nu_o & -\mu^1_1 - \mu_o \\ \mu^1_1 - \mu_o & -\nu^1_1 - \nu_o & 0 & -\rho^1_1 - \rho_o \\ \nu^1_1 - \nu_o & \mu^1_1 + \mu_o & \rho^1_1 + \rho_o & 0 \end{vmatrix} \wedge \begin{vmatrix} \theta^2 \\ \theta^2 \\ \psi^1 \\ \psi^2 \end{vmatrix} ,$$

which gives the Levi-Civita connection forms with respect to $\theta^1, \theta^2, \psi^1, \psi^2$. The curvature in this case is constant and equal to 4. (Cf. Bryant [2], p. 461). Thus $\mathbb{H}P^1$, g is isometric to the 4-sphere with its canonical metric of constant curvature equal to 4, (i.e. the sphere in $\mathbb{R}^5$ of radius equal to $\frac{1}{2}$) .

§ 11.- The twistor map.

The isomorphism (10.1) between $\mathbb{H}^{n+1}$ and $\mathbb{C}^{2n+2}$ is a $\mathbb{C}$-linear map (of right $\mathbb{C}$-modules)

$$(11.1) \qquad \tilde{T} : \mathbb{C}^{2n+2} \to \mathbb{H}^{n+1} \ , \ \begin{vmatrix} x \\ y \end{vmatrix} \to x + jy \ , \ x,y \in \mathbb{C}^{n+1} \ ,$$

which induces the twistor map

(11.2) $$T : \mathbb{C}P^{2n+1} \to \mathbb{H}P^n \quad , \quad T[{x \atop y}] = [x + jy] \quad .$$

Furthermore, the isomorphism $\tilde{T}$ induces a Lie group monomorphism

$$h : Sp(n+1) \to SU(2n+2) \quad , \quad u = A + jB \to \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} \quad ,$$

such that $\tilde{T} \circ h(u) = u \circ \tilde{T}$ for every $u \in Sp(n+1)$; i.e.

$$(A + jB)(x + jy) = Ax - \bar{B}y + j(\bar{A}y + Bx) = \tilde{T}\left(\begin{pmatrix} A & -\bar{B} \\ B & A \end{pmatrix} \begin{vmatrix} x \\ y \end{vmatrix}\right) .$$

Composing h with the natural projection

$$\pi : SU(2n+2) \to \mathbb{C}P^{2n+1} \quad , \quad \pi(u) = u \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad ,$$

we get a projection

(11.3) $$\pi_o : Sp(n+1) \to \mathbb{C}P^{2n+1} \quad , \quad \pi_o(e_o,\ldots,e_n) = [{x \atop y}] \ ,$$

where $e_o = x + jy$, $x,y \in C^{n+1}$. This is a principal $U(1) \times Sp(n)$-bundle.

The following commutative diagram is basic for the theory of minimal surfaces in $\mathbb{H}P^n$.

(11.4)
$$\begin{tikzcd} & Sp(n+1) \arrow[dl, "\pi"'] \arrow[dr, "\pi_o"] & \\ \mathbb{H}P^n & & \mathbb{C}P^{2n+1} \arrow[ll, "T"] \end{tikzcd}$$

The twistor map is a fibre bundle with standard fibre $S^2 = Sp(1)/_{U(1)}$.
We need to examine this fibration in detail.

Let $\rho : \mathbb{C}^{2n+2} \setminus \{0\} \to \mathbb{C}P^{2n+1}$ denote the canonical projection, and let $\langle v,w \rangle = {}^t\bar{v}w$ denote the standard hermitian inner product on $\mathbb{C}^{2n+2}$. For each point $v \in \mathbb{C}^{2n+2} \setminus \{0\}$ we have a complex isomorphism

$$(11.5) \qquad \rho_{*_v} : \{w \in \mathbb{C}^{2n+2} : \langle v,w \rangle = 0\} = T^{1,0}_{[v]}\mathbb{C}P^{2n+1} .$$

The Fubini-study metric on $\mathbb{C}P^{2n+1}$ is the hermitian metric induced on each $T^{1,0}_{[v]}\mathbb{C}P^{2n+1}$ by ρ_{*_v} from the standard hermitian inner product on $\mathbb{C}^{2n+1}$.

Let $[x + jy] \in \mathbb{H}P^n$, where $x,y \in \mathbb{C}^{n+1}$. Then

$$(11.6) \qquad T^{-1}\{[x + jy]\} = \left\{\left[a\begin{vmatrix} x \\ y \end{vmatrix} + b\begin{vmatrix} -\bar{y} \\ \bar{x} \end{vmatrix}\right] : a,b \in \mathbb{C}\right\} ;$$

i.e. the fibre of T through $[\begin{smallmatrix} x \\ y \end{smallmatrix}] \in \mathbb{C}P^{2n+1}$ is the line $(= \mathbb{C}P^1)$ through $[\begin{smallmatrix} x \\ y \end{smallmatrix}]$ and $[\begin{smallmatrix} -\bar{y} \\ \bar{x} \end{smallmatrix}]$.

We define the <u>horizontal distribution</u> to be the complex distribution of rank $2n$ on $\mathbb{C}P^{2n+1}$ given by : $\mathcal{D}(p)$ = the orthogonal complement to the fibre of T through $p \in \mathbb{C}P^{2n+1}$, where orthogonal complement means with respect to the Fubini-Study metric on $\mathbb{C}P^{2n+1}$.

We can use the isomorphism (11.5) to describe $\mathcal{D}$ in more detail. Consider the holomorphic 1-form on $\mathbb{C}^{2n+2}$

$$(11.7) \qquad \sigma_{\left|\begin{smallmatrix} x \\ y \end{smallmatrix}\right|} = -\,{}^ty\,dx + {}^tx\,dy \quad , \text{ where } x,y \in \mathbb{C}^{n+1} .$$

Then for any point $p = [\begin{smallmatrix} x \\ y \end{smallmatrix}] \in \mathbb{C}P^{2n+1}$,

$$(11.8) \qquad \mathcal{D}(p) = \rho_{*\left|\begin{smallmatrix} x \\ y \end{smallmatrix}\right|}\{v \in \mathbb{C}^{2n+2} : \sigma_{\left|\begin{smallmatrix} x \\ y \end{smallmatrix}\right|}(v) = 0\} .$$

Thus if $p \in U_0 = \{[{x \atop y}] \in \mathbb{C}P^{2n+1} : x^0 \neq 0\}$, we can let $s_0 : U_0 \to \mathbb{C}^{2n+2}$ be the local section defined by $s_0([{x \atop y}]) = \begin{pmatrix} x/x^0 \\ y/x^0 \end{pmatrix}$. If we put $t^a = x^a/x^0$, $s^0 = y^0/x^0$, $s^a = y^a/x^0$, then $\{t^a, s^0, s^a\}$ are local complex coordinates on U_0 and

(11.9) $$s_0^* \sigma = ds^0 + \sum_1^n t^a \, ds^a - \sum_1^n s^a \, dt^a$$

is the defining equation for $\mathcal{D}$ on U_0 . Then

$$d(s_0^*\sigma) = 2 \sum_1^n dt^a \wedge ds^a \quad , \text{ and}$$

$$d(s_0^*\sigma)^n = 2(n!) \, dt^1 \wedge ds^1 \wedge \ldots \wedge dt^n \wedge ds^n \quad , \text{ and}$$

$$(s_0^*\sigma) \wedge d(s_0^*\sigma)^n = 2(n!) \, ds^0 \wedge dt^1 \wedge ds^1 \wedge \ldots \wedge dt^n \wedge ds^n \neq 0 \; .$$

Thus the locally defined forms $s_0^* \sigma$ define a contact structure on $\mathbb{C}P^{2n+1}$, and in particular $\mathcal{D}$ does not satisfy the Frobenius condition.

Even though $\mathcal{D}$ is not completely integrable, it does have many 1-dimensional (complex) integral submanifolds.

Def. Let M be a Riemann surface. A holomorphic curve $F : M \to \mathbb{C}P^{2n+1}$ is horizontal if it is an integral submanifold of $\mathcal{D}$, i.e. $F_* \, T_p^{1,0} M \subseteq \mathcal{D}(F(p))$, for every $p \in M$; i.e. for any local lift $\tilde{F} : M \to \mathbb{C}^{2n+2} \setminus \{0\}$, $\tilde{F}^*\sigma = 0$.

An important class of holomorphic horizontal curves consists of the horizontal lines. It can be checked that if a line L in $\mathbb{C}P^{2n+1}$ is orthogonal to the fibre of T through one of its points p , then it is also orthogonal to the fibre of T through every one of its points.

Bryant [2] has characterized the holomorphic horizontal curves in $\mathbb{C}P^3$.

Theorem 11.1.-

Let M be a connected Riemann surface, and let f and g be meromorphic functions on M with g non-constant. Then $F(f,g) : M \to \mathbb{C}P^3$ defined by

$$F(f,g) = \begin{bmatrix} 1 \\ g \\ f - \frac{1}{2}\, g\, \frac{df}{dg} \\ \frac{1}{2}\frac{df}{dg} \end{bmatrix}$$

is a holomorphic horizontal curve in $\mathbb{C}P^3$. Conversely, any non constant holomorphic horizontal curve $F : M \to \mathbb{C}P^3$ is either of the form $F = F(f,g)$ for some unique meromorphic functions f and g on M , or $F(M)$ is contained in some line in $\mathbb{C}P^3$.

Remark on notation : (Cf. Griffiths and Harris [10]).

A holomorphic map $F : M \to \mathbb{C}P^n$ from a Riemann surface M is given by $n+1$ meromorphic functions :

$$F = \begin{bmatrix} f^0 \\ \vdots \\ f^n \end{bmatrix} ,$$

defined only up to their ratios. A meromorphic function f on M is a map $f : M \to \mathbb{C} \cup \{\infty\}$ of M into the Riemann sphere which is holomorphic at any point $p \in M$ where $f(p)$ is finite ; i.e. $f(p) \in \mathbb{C}$; and for which $1/f$ is holomorphic at any point p where $f(p) = \infty$.

If f and g are meromorphic functions on M , and g is non-constant, then $\frac{df}{dg}$ is the meromorphic function on M defined as follows. Let z be a local complex coordinate in M . Then $df = f'(z)\,dz$ and $dg = g'(z)\,dz$, where f' and g' are holomorphic functions of z , and $g' \neq 0$. Then $\frac{df}{dg} = \frac{f'(z)}{g'(z)}$, which is meromorphic on the domain of z , and is independent of the choice of complex coordinate z .

Proof of Theorem 11.1. :

We use the notation just preceding (11.9), for the case $n = 1$. Rewriting (11.9) we get on U_0 ,

$$\sigma_0 = s_0^*\sigma = ds^0 + t^1 ds^1 - s^1 dt^1 = d(s^0 + t^1 s^1) - 2s^1 dt^1 .$$

A simple calculation verifies that $F(f,g)^* \sigma_0 = 0$, and thus that $F(f,g)$ is horizontal on $F(f,g)^{-1} U_0 \subset M$. But this is an open dense subset of M , and $\mathcal{D}$ is a smooth distribution on $\mathbb{C}P^3$. Hence $F(f,g)$ is horizontal.

Conversely, suppose that $F : M \to \mathbb{C}P^3$ is a horizontal holomorphic curve. Suppose first of all that $F(M) \cap U_0 \neq \emptyset$.

Then $s_0 \circ F = \begin{vmatrix} 1 \\ t^1 \\ s^0 \\ s^1 \end{vmatrix}$, where we write $t^1 = t^1 \circ F$, etc, meromorphic functions on M .

If $t^1 = \text{constant} = a$, say, then F horizontal implies that $0 = F^* \sigma_0 = d(s^0 + as^1)$. Thus $s^0 + as^1 = \text{constant} = b$, say, and $F(M) \subset \left\{ \begin{bmatrix} 1 \\ a \\ b-as \\ s \end{bmatrix} : s \in \mathbb{C} \right\}$, which is a line in $\mathbb{C}P^3$.

On the other hand, if t^1 is not constant, put $f = s^0 + t^1 s^1$ and $g = t^1$. Then $0 = F^* \sigma_0 = df - 2s^1 dg$ implies that $s^1 = \frac{1}{2}\frac{df}{dg}$, and consequently that $s^0 = f - \frac{1}{2}\, g\, \frac{df}{dg}$. Hence $F = F(f,g)$.

We are left with the possibility that $F(M) \cap U_o = \emptyset$; i.e. that $F(M) \subset \{[{x \atop y}] : x^o = 0\}$. Suppose that $F(M)$ is not contained in the line $\{[{x \atop y}] : x = 0\}$. Let $U_1 = \{[{x \atop y}] : x^1 \neq 0\}$, and put $s_1 : U_1 \to \mathbb{C}^4$, $s_1[{x \atop y}] = \begin{bmatrix} r^o \\ 1 \\ r^2 \\ r^3 \end{bmatrix}$, where $r^o = \dfrac{x^o}{x^1}$, $r^2 = \dfrac{y^o}{x^1}$, $r^3 = \dfrac{y^1}{x^1}$. Then as above we write $s_1 \circ F = \begin{bmatrix} 0 \\ 1 \\ r^2 \\ r^3 \end{bmatrix}$, where $r^2 = r^2 \circ F$, $r^3 = r^3 \circ F$ are meromorphic functions on M . Then $0 = (s_1 \circ F)^* \sigma = dr^3$ implies that r^3 is contant = C , say. Hence $F(M)$ is contained in the line $\left\{ \begin{bmatrix} o \\ 1 \\ c \\ r \end{bmatrix} : r \in \mathbb{C} \right\}$. Thus in every case we have shown that either $F = F(f,g)$ or $F(M)$ is contained in a line.

The following theorem is the main result in Bryant [2] . We shall give here only a sketch of the proof.

Theorem 11.2.-

Let M be a compact connected Riemann surface. Then there exists a holomorphic horizontal imbedding of M into $\mathbb{C}P^3$.

Sketch of proof :

It is a consequence of the Riemann-Roch theorem that for any compact Riemann surface M there exists a generically 1:1 immersion $F : M \to \mathbb{C}P^2$ whose only singularities are ordinary double points. (Cf. Griffiths-Harris [10]). Such a map is given by a pair of meromorphic functions on M , but they are determined only up to a projective transformation of $\mathbb{C}P^2$. That is, if $F(M)$ is not contained in a line, we have $F = \begin{bmatrix} 1 \\ g \\ f \end{bmatrix}$, where g and f are meromorphic functions on M . If $T \in PGL(3;\mathbb{C})$ is a projective transformation, then $T \circ F = \begin{bmatrix} 1 \\ \tilde{g} \\ \tilde{f} \end{bmatrix}$, and $T \circ F$ has the same properties as F . The idea of Bryant's proof is to choose T in such a way that $F(\tilde{f},\tilde{g}) : M \to \mathbb{C}P^3$ is an imbedding.

To do this let $Q_0 \in \mathbb{C}P^2$ be a point which does not lie on $C = F(M)$, any flex tangent or bitangent to C , or the tangent cone to any double point of C . Let L be a line through Q_0 which is not tangent to C nor does it pass through any double point of C . Let $Q_1 \in L$, distinct from Q_0 and not on C . (See figure 1).

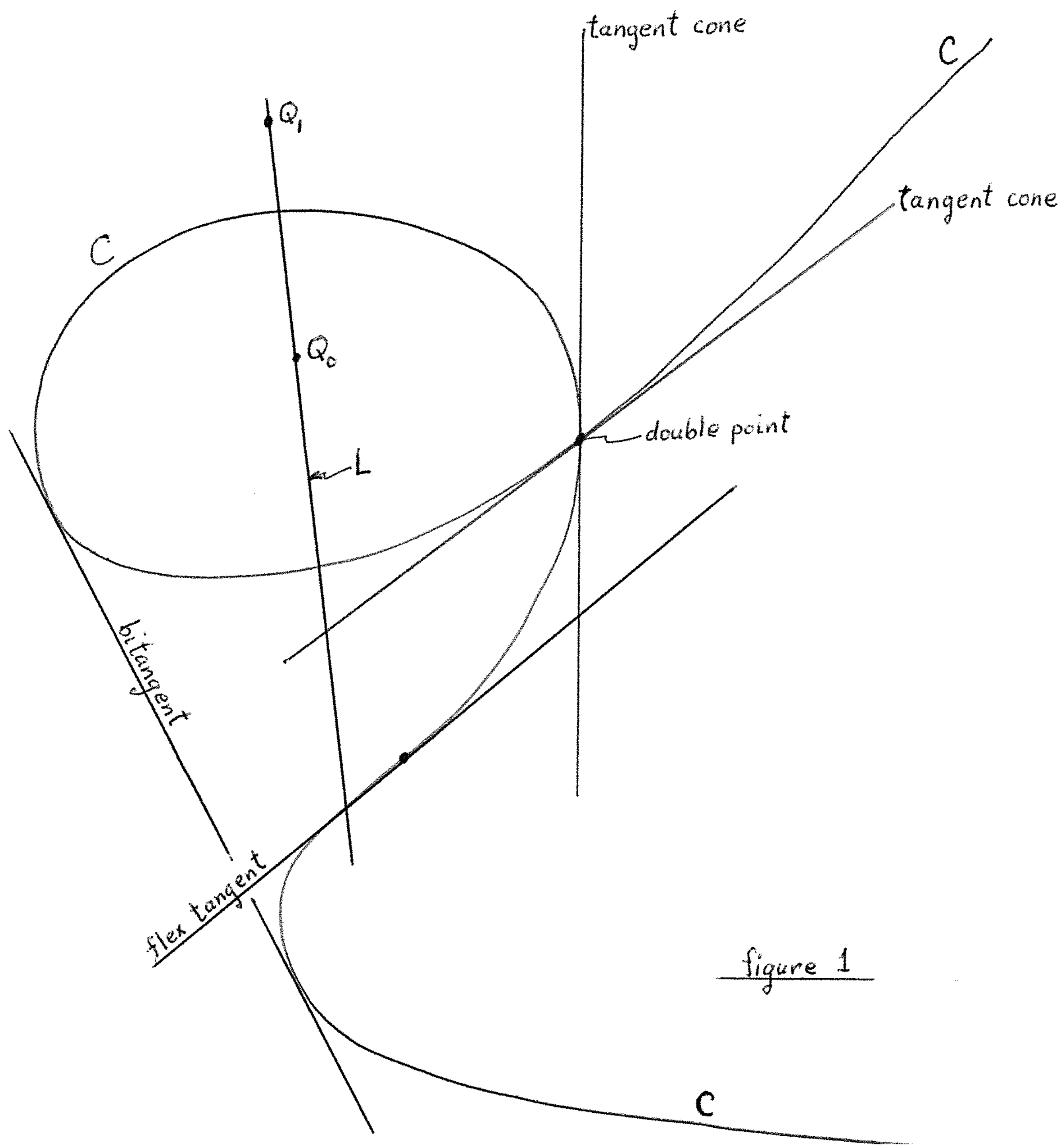

figure 1

Choose $T \in PGL(3;\mathbb{C})$ such that $T\begin{bmatrix}0\\1\\0\end{bmatrix} = Q_0$ and $T\begin{bmatrix}0\\0\\1\end{bmatrix} = Q_1$. If we replace F by $T^{-1} \circ F$, then we replace Q_0 by $T^{-1}Q_0 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ and Q_1 by $T^{-1}Q_1 = \begin{bmatrix}0\\0\\1\end{bmatrix}$. Setting $T^{-1} \circ F = \begin{bmatrix}1\\g\\f\end{bmatrix}$, one can then show that $F(f,g) : M \to \mathbb{C}P^3$ is an imbedding.

Examples : The "W-curves" .

Consider holomorphic maps $F : \mathbb{C} \cup \{\infty\} \to \mathbb{C}P^3$ given by $F(z) = \begin{bmatrix}1\\az^m\\bz^n\\cz^p\end{bmatrix}$, where m,n,p are integers, a,b,c are complex constants. Then F is horizontal iff

$$n = m + p \quad \text{and} \quad b(m+p) = ac(m-p) \ .$$

To avoid repetitions it suffices to consider $p > m > 0$. Omitting lines and considering only immersions, we consider only the cases $m = 1$, $p > 1$. Thus

$$F(z) = \begin{bmatrix}1\\az\\bz^{1+p}\\cz^p\end{bmatrix} \ , \text{ where } \quad b = a\,c\,\frac{(1-p)}{1+p} \ .$$

For fixed $p > 1$, $F(z)$ is a holomorphic horizontal imbedding of S^2 into $\mathbb{C}P^3$ for every choice of the parameters $a,c \in \mathbb{C} \setminus \{0\}$, ($c = 0$ is a line). The case $p = 2$, $a = \sqrt{3}$, $c = -\sqrt{3}$ is the Veronese surface in $\mathbb{C}P^3$.

§ 12.- Minimal surfaces in $\mathbb{H}P^n$.

Let M, ds^2 be an oriented connected surface with a Riemannian metric ds^2 , and let $f : M \to \mathbb{H}P^n$ be an isometric immersion. We use the notation of sections 10 and 11.

A local symplectic frame field along f is a map $u : U \to Sp(n+1)$, where U is an open subset of M, such that $\pi \circ u = f$, where $\pi : Sp(n+1) \to \mathbb{H}P^n$ is the projection of § 10.

Since f is an immersion, any local symplectic frame along f is locally the composition with f of a local symplectic frame field in $\mathbb{H}P^n$.

We can write $u = (e_o, e_1, \dots, e_n)$, where $e_A : U \to \mathbb{H}^{n+1}$ is the $A^{\underline{th}}$ column of u. Then $\pi \circ u = f$ means that $\rho \circ e_o = f$, where $\rho : \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{H}P^n$ is the natural projection of § 10. For $n > 1$ there is in general no symplectic frame field along f which is a Darboux frame. (Compare to the situation in $\mathbb{C}P^n$, see Chern-Wolfson [6] and Eschenburg-Tribujy-Quadaloupe [9]). We shall construct the <u>first order</u> symplectic frames.

If we put $\varphi^a = u^* \Phi^a_o$, then

(12.1) $$de_o = e_o\, u^* \Phi^o_o + e_a\, \varphi^a \quad .$$

Thus $f_* = \rho_{*_{e_o}} \circ de_o = (\rho_{*_{e_o}} e_a)\, \varphi^a$.

Let θ^1, θ^2 be an oriented orthonormal coframe field in $U \subset M$, so that $ds^2 = (\theta^1)^2 + (\theta^2)^2$ on U. Then

(12.2) $$\varphi = (\varphi^a) = p\theta^1 + q\theta^2 \quad , \text{where}$$

$p,q : U \to \mathbb{H}^n$ are smooth maps. It follows that

(12.3) $$f_* = (\rho_{*_{e_o}} e_a)(p^a\theta^1 + q^a\theta^2)\ , \text{ where } p = (p^a)\ ,\ q = (q^a)\ .$$

The condition that f be isometric is that

$$\mathrm{Re}\ {}^t\overline{(p\theta^1 + q\theta^2)}(p\theta^1 + q\theta^2) = (\theta^1)^2 + (\theta^2)^2 \quad , \text{ which is}$$

$$(12.4) \qquad |p|^2 = 1 = |q|^2 \quad \text{and} \quad \mathrm{Re}(p,q) = 0 \ .$$

Thus associated to the symplectic frame field u we have a map

(12.5) $\{p,q\} : U \to G_2(\mathbb{H}^n)$ = the space of real 2-dimensional subspaces of $\mathbb{H}^n$.

Any other symplectic frame field along f on U is given by

$$(12.6) \qquad \tilde{u} = u\,K \ , \quad K = \begin{vmatrix} a & 0 \\ 0 & A \end{vmatrix} \ , \quad a : U \to Sp(1) \ , \quad A : U \to Sp(n) \ .$$

Since $\tilde{u}^* \Phi_0 = \mathrm{Ad}(K^{-1})\, u^* \Phi_0$, we have from (10.8) that

$$(12.7) \qquad \tilde{p} = A^{-1}pa \quad , \quad \tilde{q} = A^{-1}qa \quad ,$$

where $\quad \tilde{\varphi} = \tilde{u}^*(\Phi_0^a) = \tilde{p}\theta^1 + \tilde{q}\theta^2 \quad .$

We must determine the orbit structure of the action of $Sp(1) \times Sp(n)$ on $G_2(\mathbb{H}^n)$, induced from the $\mathbb{R}$-linear action (10.8) of $Sp(1) \times Sp(n)$ on $\mathbb{H}^n$, namely of

$$(12.8) \qquad (a,A)\,\{p,q\} = \{Apa^{-1} \ , \ Aqa^{-1}\} \quad .$$

In these notes we will consider only the case $n = 1$. We refer to Jensen [12] for the details of the general case. One has the following result.

Theorem 12.1.-

A slice for the action (12.8) on $G_2(\mathbb{H}^n)$, for $n \geq 2$, is

$$\left\{ \left\{ \begin{vmatrix} \cos\alpha \\ -\sin\alpha \\ 0 \\ \vdots \\ 0 \end{vmatrix} \ , \quad i \begin{vmatrix} \cos\alpha \\ \sin\alpha \\ 0 \\ \vdots \\ 0 \end{vmatrix} \right\} : 0 \leq \alpha \leq \frac{\pi}{4} \right\} \ , \quad i = \sqrt{-1} \quad .$$

The function α is a first order invariant. The values $\alpha = 0$ and $\alpha = \frac{\pi}{4}$ are the singular orbits.

A symplectic frame u is said to be of <u>first order</u> if its associated map (12.5) takes values in the above slice ; i.e.

$$\varphi^1 = \cos \alpha(\theta^1 + i\theta^2)$$

$$\varphi^2 = \sin \alpha(\theta^1 - i\theta^2) \qquad \text{and} \quad \varphi^\alpha = 0 \quad , \quad \alpha \geq 3 \ .$$

The invariant α is zero at a point $p \in M$ means that the tangent space $f_* T_p M$ is contained in a quaternionic line. In general, $\alpha(p)$ is the angle between $f_* T_p M$ and the "best fitting" quaternionic line.

<u>The case</u> $n = 1$.

In our previous index convention, the indices a,b,c now take only the value 1 , and consequently we will usually omit them.

Now the action (12.7) of $Sp(1) \times Sp(1)$ on $\mathbb{H} \cong \mathbb{R}^4$ is just the linear representation of $Sp(1) \times Sp(1)$ which gives the standard covering projection of this group onto $SO(4)$. It follows that this action is transitive on $G_2(\mathbb{H})$, and we can take the point $\{1,i\} \in G_2(\mathbb{H})$ as the slice $(i = \sqrt{-1})$.

<u>Def</u>. A symplectic frame $u = (e_0, e_1)$ is of <u>first order</u> if

$$\varphi = \theta^1 + i\,\theta^2 \ , \tag{12.9}$$

where θ^1, θ^2 is an oriented orthonormal coframe field in M .

Smooth first order symplectic frames exist locally about any point of M .

Proposition 12.1.-

The isotropy subgroup of the action (12.8) at the point $\{1,i\}$ is $G_1 + j\,G_1$, where

$$G_1 = \left\{\begin{pmatrix} a & o \\ o & b \end{pmatrix} \;:\; a,b \in U(1)\right\} \qquad (U(1) = \{z \in \mathbb{C} : |z| = 1\}) .$$

Proof.

Let $a,b \in Sp(1)$ and suppose that $(a,b)\{1,i\} = \{1,i\}$. This means that $ba^{-1} = \cos t + i \sin t = e^{it}$ and $bia^{-1} = -\sin t + i \cos t = i\,e^{it}$ for some $t \in \mathbb{R}$, if (a,b) preserves the orientation of $\{1,i\}$. If orientation is reserved then $ba^{-1} = e^{-it}$ and $bia^{-1} = -\,i\,e^{-it}$.

Putting $a = a_1 + j\,a_2$, $b = b_1 + j\,b_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{C}$, we derive from the above equations that $a_2 = 0 = b_2$, in the orientation preserving case ; and $a_1 = 0 = b_1$ in the orientation reserving case.

The second fundamental forms.

Let $u : U \subset M \to Sp(2)$, $u = (e_0, e_1)$, be a local first order frame along f. The notation from § 10 becomes in the case $n = 1$ (cf. (10. 4) and (10.10))

$$\text{(12.10)} \qquad u^* \Phi = \begin{vmatrix} i\rho_0 + j\,\beta_0^0 & -\,\bar{\alpha} + j\,\beta \\ \alpha + j\,\beta & i\rho_1 + j\,\beta_1^1 \end{vmatrix} ,$$

where ρ_0 and ρ_1 are real. Then u of first order means that

$$\beta = 0 \quad \text{and} \quad \alpha = \varphi = \theta^1 + i\theta^2 ,$$

where θ^1, θ^2 is an oriented orthonormal coframe on U.

Using the notation of (10.8) we have

$$\beta = \psi^1 + i\,\psi^2 = 0 \tag{12.11}$$

and thus u is a Darboux frame field along f.

To find the second fundamental forms of f with respect to this Darboux frame, we need to take the exterior differential of (12.11) and use (10.6) :

$$0 = d\beta = du^*\beta^1_0 = -\beta^1_1 \wedge \alpha + \beta^0_0 \wedge \bar{\alpha} \quad .$$

Hence

$$\begin{aligned} \beta^1_1 &= a\alpha + b\,\bar{\alpha} \\ -\beta^0_0 &= b\alpha + c\,\bar{\alpha} \end{aligned} \tag{12.12}$$

, where a,b,c are complex valued smooth functions on U.

We define a complex valued symmetric bilinear form on $U \subset M$ by

$$II = \beta^1_1\alpha + (-\beta^0_0)\bar{\alpha} = a\alpha^2 + 2b\,\alpha\bar{\alpha} + c\,\bar{\alpha}^2 \qquad \text{(symmetric product)}. \tag{12.13}$$

<u>Proposition 12.2</u>.-

Let II^3 and II^4 denote the second fundamental forms of f with respect to the Darboux coframe $\theta^1, \theta^2, \psi^1, \psi^2$. Then

$$II = II^3 + i\,II^4 \quad .$$

Proof.

The Levi-Civita connection forms with respect to this coframe field are given by (10.12) . Thus, by (2.5),

$$II^3 = (\mu_1^1 - \mu_0)\theta^1 - (\nu_1^1 + \nu_0)\theta^2$$

$$II^4 = (\nu_1^1 - \nu_0)\theta^1 + (\mu_1^1 + \mu_0)\theta^2 \ .$$

Hence

$$II^3 + i\ II^4 = (\beta_1^1 - \beta_0^0)\theta^1 + i(\beta_1^1 + \beta_0^0)\theta^2 = II \ .$$

Corollary 2.1.-

The minimal surface equations for $f : M \to \mathbb{H}P^1$ are $b = 0$, where b is the complex function defined by (12.12).

Proof. $II = II^3 + i\ II^4 = (h_{ij}^3 + i\ h_{ij}^4)\ \theta^i\theta^j$, where we use the notation of (2.5). If we expand out (12.13) using $\alpha = \theta^1 + i\ \theta^2$ and then compare the coefficients of $\theta^i\theta^j$ in these two expressions for II, we get that

$$h^3 + i\ h^4 = \begin{vmatrix} a + 2b + c & i(a - c) \\ i(a - c) & - a + 2b - c \end{vmatrix} \ , \tag{12.14}$$

where $h^3 = (h_{ij}^3)$, $h^4 = (h_{ij}^4)$ are 2×2 symmetric matrices.

Hence Trace h^3 + i Trace $h^4 = 4b$, and the corollary follows immediately.

Corollary 2.2.-

If f is minimal then

$$a = \frac{1}{2}\ (L(h^3) + iL(h^4))$$

$$c = \frac{1}{2}\left(\overline{L(h^3)} + i\overline{L(h^4)}\right) \quad , \text{ and hence}$$

$$a\bar{c} = \frac{1}{4}\left(L(h^3)^2 + L(h^4)^2\right) \quad ,$$

where L is Hopf's transformation (3.1).

Proof.

From (12.14), if $b = 0$, we have

$$h^3_{11} + i\, h^4_{11} = a + c$$

$$h^3_{12} + i\, h^4_{12} = i(a - c) \ .$$

Solving for a and c, one obtains the formulas of the Corollary.

Change of first order frame.-

If $\tilde{u} : \tilde{U} \to Sp(2)$, $\tilde{u} = (\tilde{e}_0, \tilde{e}_1)$ is any other first order frame field along f, then by Proposition 12.1 on $U \cap \tilde{U}$,

(12.15) $$\tilde{u} = u \begin{pmatrix} e^{ir} & 0 \\ 0 & e^{is} \end{pmatrix}$$, where $r, s : U \cap \tilde{U} \to \mathbb{R}$ are smooth functions, (because orientation must be preserved).

Lemma 12.1.

$$\tilde{\alpha} = e^{i(r-s)}\, \alpha$$

$$\tilde{\beta}^0_0 = e^{2ir}\ \beta^0_0$$

$$\tilde{\beta}^1_1 = e^{2is}\ \beta^1_1 \ .$$

Proof. $$\begin{vmatrix} i\,\tilde{\rho}_0 + j\,\tilde{\beta}^0_0 & -\bar{\tilde{\alpha}} \\ \tilde{\alpha} & i\,\tilde{\rho}_1 + j\,\tilde{\beta}^1_1 \end{vmatrix} = \tilde{u}^*\,\Phi = \tilde{u}^{-1}\ d\tilde{u} =$$

$$\begin{vmatrix} e^{-ir} & o \\ o & e^{-is} \end{vmatrix} \; u^* \Phi \begin{vmatrix} e^{ir} & o \\ o & e^{is} \end{vmatrix} + \begin{vmatrix} idr & o \\ o & ids \end{vmatrix}$$, from which

the Lemma follows.

The quartic form.-

It follows from Lemma 12.1 that the quartic form

$$(12.16) \qquad Q = \overline{\beta^o_o} \;\; \beta^1_1 \; \alpha\alpha \qquad \text{(symmetric product)}$$

is globally defined on M . Moreover, if f is minimal, then formulas (12.12) shows that

$$(12.17) \qquad Q = - a\bar{c} \;\; \alpha^4 \quad ,$$

which shows that Q is a bidegree $(4,0)$ form with respect to the complex structure $\alpha = \theta^1 + i\,\theta^2$ defined on M by its Riemannian metric. Furthermore Hopf's remarkable observation of section 3 continues to bear fruit.

Theorem 12.2.- If $f : M \to \mathbb{H}P^1$ is minimal then Q is holomorphic.

Proof.

Use the proof of Hopf's theorem in § 3 as a model and use the structure equations (10.6) of $Sp(2)$.

Bryant has defined a minimal surface $f : M \to \mathbb{H}P^1$ to be superminimal if Q is identically zero on M . By the Riemann-Roch theorem any holomorphic quartic form on S^2 must be zero. Hence minimal S^2 in $\mathbb{H}P^1 = S^4$ is necessarily superminimal.

If f is superminimal, then $a\bar{c} = 0$ on U from (12.17). As a result of the following lemma we see that a and c are functions with the property that they are either identically zero or else they have isolated zeros. It follows then that for a superminimal surface either

i) $\beta_0^0 \equiv 0$ for any first order frame field ; or

ii) $\beta_1^1 \equiv 0$ for any first order frame field.

Bryant has called the two possible cases i) and ii) positive spin and negative spin, respectively.

Lemma 12.1.-

Suppose $f : M \to \mathbb{H}P^1$ is minimal. Let a and c be the functions defined in (12.12) with respect to some first order frame u on $U \subset M$. Then

$$(d\bar{c} - i\bar{c}(\rho_1^1 - 3\rho_0)) \wedge \alpha = 0$$

$$(da - ia(3\rho_1^1 - \rho_0)) \wedge \alpha = 0 .$$

In particular, if z is any local complex coordinate about a point of U, then

$$\frac{\partial \bar{c}}{\partial \bar{z}} = h\bar{c} \quad , \quad \frac{\partial a}{\partial \bar{z}} = ka \quad ,$$

where h and k are smooth functions on the domain of z .

Proof.

Take the exterior differential of (12.12) and use the structure equations (10.6) . We have used the notation of (12.10).

It follows from Chern's Theorem (4.1) that a and c each has isolated zeros or vanishes identically.

Remarks

1. The functions a and c are both identically zero iff f is totally geodesic. In fact, from Corollary 2.2, a and c are both zero iff $L(h^3) \equiv 0 \equiv L(h^4)$; i.e. $II^3 \equiv 0 \equiv II^4$.

2. Suppose $a\bar{c} = 0$, but not both a and c are identically zero. Then from Corollary 2.2, $L(h^4) = \ i\ L(h^3)$, where $\varepsilon = \pm 1$ is constant on M . Thus

$$a = \frac{1}{2}(1 - \varepsilon)\ L(h^3)$$

$$\bar{c} = \frac{1}{2}(1 + \varepsilon)\ L(h^3) \ .$$

Chern [4] points out that a reversal of the orientation of the normal bundle of M in S^4 will reverse the sign of ε . In fact if h^3 , h^4 are the matrices of the second fundamental forms of f with respect to a Darboux frame e_1 , e_2 , e_3 , e_4 , then h^3 , $- h^4$ are the matrices with respect to e_1 , e_2 , e_3 , $- e_4$.

Bryant [2] points out that if $u = (e_0, e_1)$ is a first order frame along f , then the associate map $f_1 : M \to \mathbb{H}P^1$ defined by $f_1 = [e_1]$ is globally defined (does not depend on choice of first order frame). Furthermore $u_1 = (e_1, e_0)$ is a first order frame along f_1 , as from (12.10) one gets

$$u_1^* \Phi = \begin{vmatrix} i\rho_1 + j\beta_1^1 & \alpha \\ -\bar{\alpha} & i\rho_0 + j\beta_0^0 \end{vmatrix} \ .$$

Hence, denoting the forms with respect to u_1 with tildes, one has

$$\tilde{\alpha} = -\bar{\alpha} \quad , \quad \tilde{\beta}_0^0 = \beta_1^1 \quad , \quad \tilde{\beta}_1^1 = \beta_0^0 \quad .$$

Consequently, f_1 is also a superminimal isometric immersion and $\tilde{a} = c$, $\tilde{c} = a$; i.e. the spin of f_1 is the opposite of the spin of f . (Of course u_1 has reversed the orientation of M , but then $\tilde{u}_1 = (e_1 j , e_0 j)$ will preserve the orientation of M . It is easily checked that the spin of f does not change even under a change of first order frame which reverses orientation of M) .

§ 13.- The directrix curve.

We now tie together sections 11 and 12. The diagram (11.4) plays an essential role in this discussion. Let $f : M \to \mathbb{H}P^1$ be an isometric immersion. If $u : U \to Sp(2)$ is any first order symplectic frame along f , then the map $\pi_0 \circ u : U \to CP^3$ does not depend on the choice of u , and hence is globally defined (where π_0 is the projection in (11.4)). In fact any other first order frame $\tilde{u} : \tilde{U} \to Sp(2)$ is related to u on $U \cap \tilde{U}$ by (12.15) , and hence $\pi_0 \circ \tilde{u} = \pi_0 \circ u$.

Definition.-

The smooth map $F : M \to CP^3$ defined locally by $F = \pi_0 \circ u$ for any first order frame u , is called the directrix curve of f . Clearly the twistor projection of F is $f : T \circ F = f$.

(13.1) Put $u = (e_0, e_1)$ and $e_0 = x + jy$, $e_1 = z + jw$, where $x,y,z,w : U \to \mathbb{C}^2$ are smooth maps. Then

$$\pi_0 \circ u = [{}^{x}_{y}] .$$

Put $\widetilde{F} = \begin{vmatrix} x \\ y \end{vmatrix} : U \to \mathbb{C}^4$. From (12.10) and the fact that u is of first order we have

$$d\widetilde{F} = \begin{vmatrix} x \\ y \end{vmatrix} i\rho_0 + \begin{vmatrix} -\bar{y} \\ \bar{x} \end{vmatrix} \beta^0_0 + \begin{vmatrix} z \\ w \end{vmatrix} \alpha .$$

Hence,

$$(13.2) \qquad F_* = \rho_{*\begin{vmatrix} x \\ y \end{vmatrix}} \circ d\widetilde{F} = \rho_{*\begin{vmatrix} x \\ y \end{vmatrix}} \begin{vmatrix} -\bar{y} \\ \bar{x} \end{vmatrix} \beta^0_0 + \rho_{*\begin{vmatrix} x \\ y \end{vmatrix}} \begin{vmatrix} z \\ w \end{vmatrix} \alpha ,$$

where $\rho : \mathbb{C}^4 - \{0\} \to \mathbb{C}P^3$ is the projection. Since α is a bidegree (1,0) form on M , we have the following.

Theorem 13.1.-

f is superminimal with positive spin iff its directrix curve F is a horizontal, holomorphic immersion.

Remark.

f is superminimal with negative spin iff its associate surface f_1 is superminimal with positive spin. The directrix curve of f_1 is F_1 defined locally by $\begin{bmatrix} zj \\ wj \end{bmatrix}$, since $(e_1 j , e_0 j)$ is a first order frame along f_1 .

Theorem 13.2.-

Any holomorphic, horizontal immersion $F : M \to \mathbb{C}P^3$ is the directrix curve of its twistor projection $f = T \circ F$ (which therefore is superminimal with positive spin).

Proof.

We shall actually prove a more general result, which we will state at the end. Let $F : M \to \mathbb{C}P^3$ be an immersion such that the twistor projection $f = T \circ F$ is an isometric immersion ; (i.e. F is transversal to the fibres of F and the metric on M is given by $ds^2 = f^*g$) .

We define a local symplectic frame field along F on an open set $U \subset M$ to be a smooth map $u : U \to Sp(2)$ such that $\pi_0 \circ u = F$.

It is evident that any symplectic frame field u along F is also one along its twistor projection $f = T \circ F$. But the converse is not true. In fact, any other frame field $\tilde{u} : U \to Sp(2)$ along F must be related to u by

$$(13.3) \qquad \tilde{u} = u \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} ,$$

where $A : U \to U(1)$ and $B : U \to Sp(1)$. We set $B = B_1 + jB_2$, where $B_1, B_2 : U \to \mathbb{C}$ satisfy $|B_1|^2 + |B_2|^2 = 1$.

If we put $u^* \Phi_0^1 = \alpha + j\beta$ and $\tilde{u}^* \Phi_0^1 = \tilde{\alpha} + j\tilde{\beta}$, as in (12.10), then by (10.8) we have

$$(13.4) \qquad \begin{aligned} \tilde{\alpha} &= (B_1\alpha - \bar{B}_2\beta) A^{-1} \\ \tilde{\beta} &= (B_2\alpha + B_1\beta) A^{-1} \end{aligned} .$$

Using the notation of (13.1) we obtain a more general version of (13.2) :

(13.5) $$F_* = \rho_{*_{\left|\begin{matrix} x \\ y \end{matrix}\right|}} \left|\begin{matrix} -\bar{y} \\ \bar{x} \end{matrix}\right| \beta^0_0 + \rho_* \left|\begin{matrix} z \\ w \end{matrix}\right| \alpha + \rho_* \left|\begin{matrix} -\bar{w} \\ \bar{z} \end{matrix}\right| \beta \quad ,$$

where we use the notation of (12.10) : $u^* \; \Phi^0_0 = i\rho_0 + j\beta^0_0$.

(13.6) <u>Assumption</u> : Suppose that F has the property that α and β are bidegree (1,0) forms on M for any symplectic frame fields along F . For example, this assumption would be satisfied if F were holomorphic.

Under this assumption we have

(13.7) $$\alpha = p\,\varphi \quad , \quad \beta = q\,\varphi \quad ,$$

where $\varphi = \theta^1 + i\theta^2$, θ^1 , θ^2 an oriented orthonormal coframe field on M ; and $p,q : U \to \mathbb{C}$ satisfy

(13.8) $$|p|^2 + |q|^2 = 1 \quad ,$$

since $f_* = T_*\rho_* \binom{z}{w}\alpha + T_*\rho_* \begin{pmatrix} -\bar{w} \\ \bar{z} \end{pmatrix} \beta$ is an isometry.

Expressing the same quantities with respect to $\tilde{u}$ with a tilde, we have from (13.4)

(13.9) $$\begin{aligned} \tilde{p} &= (B_1 p - \bar{B}_2 q)\, A^{-1} \\ \tilde{q} &= (B_2 p + \bar{B}_1 q)\, A^{-1} \quad . \end{aligned}$$

If we take $B_1 = \bar{p}$, $B_2 = -q$, $A = 1$, then from (13.8) we have that $B = B_1 + jB_2 \in Sp(1)$, and from (13.9) we have $\tilde{q} = 0$; i.e. $\tilde{\beta} = 0$.

Hence, for the kind of maps $F : M \to \mathbb{C}P^3$ considered here, there exist on a neighborhood of any point of M , symplectic frame fields u with respect to which $\beta = 0$; i.e. frame fields which are of first order along the twistor projection $f = T \circ F$. We shall call these first order frames along F .

Now suppose that F is holomorphic and horizontal. Let u be a local first order frame along F . Then u is a first order frame along $f = T \circ F$, and by (13.5) $\beta_0^0 = 0$ because F is horizontal. Hence f is superminimal with positive spin. Furthermore, $\pi_0 \circ u = F$, so that F is the directrix curve of f . This prove Theorem 13.2.

Eells and Salamon [8] have characterized the directrix curve of any minimal surface in $\mathbb{H}P^1$ as follows. Let J_2 denote the (non-integrable) almost complex structure on the tangent space at any point to be the usual one on the horizontal subspace, and the negative of the usual one on the vertical subspace.

Referring to (13.5) one sees that a map $F : M \to \mathbb{C}P^3$ is J_2-holomorphic iff α and β are of bidegree (1,0) and β_0^0 is of bidegree (0,1) on $U < M$. Thus any J_2-holomorphic map whose twistor projection f is an isometric immersion satisfies the assumption (13.6), and thus possesses first order frames. If u is such a frame field, then $\beta_0^0 = -b\alpha - c\bar{\alpha}$ is of bidegree (0,1) implies that $b = 0$. Hence f is minimal. The converse is clear, and we have proved :

Theorem 13.3.-

1) If $f : M \to \mathbb{H}P^1$ is a minimal isometric immersion, then its directix curve $F : M \to \mathbb{C}P^3$ is J_2-holomorphic.

2) If $F : M \to \mathbb{C}P^3$ is J_2-holomorphic and transverse to the twistor map T, on a Riemann surface M, then its twistor projection $f = T \circ F : M \to \mathbb{H}P^1$ is a minimal conformal immersion.

Remarks :

1. There exist many minimal surfaces in $\mathbb{H}P^1 = S^4$ which are not superminimal. In fact, there are the well known examples of Lawson [13] of minimal immersions of any compact surface (except $\mathbb{R}P^2$) into the 3 sphere S^3, none of which is totally geodesic except $S^2 \to S^3$. But if $f : M \to S^4$ is superminimal and $f(M) \subset S^3 \subset S^4$, where $S^3 \subset S^4$ as the "equator"), then $f(M)$ must be totally geodesic. This is true because one can choose a local Darboux frame field such that the fourth vector is parallel (the normal to S^3). Thus $h^4 \equiv 0$ (see (2.5)), and for a superminimal surface $L(h^3) = i\, L(h^4)$, so that $h^3 \equiv 0$ also ; i.e. f is totally geodesic.

2. Theorem 11.2 and Theorem 13.2 combined give Bryant's result [2] : Any compact Riemann surface M can be conformally immersed in $S^4 = \mathbb{H}P^1$.

3. A "Weierstrass formula", of the type of Theorem 11.1, for transversal J_2-holomorphic maps of M into $\mathbb{C}P^3$ would be very interesting.

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