1. Manifolds

M, N, X, Y. Local coords $(x^1,..,x^n)$ or just (x^i), $x^i: U \subseteq M \to \mathbb{R}$ fns. $i=1,..,n=\dim M$

Tangent space at $p \in M$ is T_pM, tangent bundle $TM = \bigcup_{p\in M} T_pM$

Tangent vector (p,v), $p \in M$, $v \in T_pM$. TM locally a product.

Function $f: M \to \mathbb{R}$ gives $df_{(p)}: T_pM \to \mathbb{R}$, $df(v) = v(f)$

$(x^1,..,x^n)$ local coords in M, then $(x^1,..,x^n, \dot{x}^1,..,\dot{x}^n)$ denote the local coords in TM, $(x^i, \dot{x}^i)(p,v) = (x^i(p), dx^i(v))$

so $v = \dot{x}^i(p,v)\frac{\partial}{\partial x^i}(p)$ summation convention always.

Note: $df = \frac{\partial f}{\partial x^i} dx^i$ encodes partial derivatives.

Cotangent space $T_p^*M = (T_pM)^* = \mathrm{Hom}(T_pM, \mathbb{R})$.

bundle $T^*M = \bigcup_{p\in M} T_p^*M$.

Local coords $(x^i, \dot{x}_i)$ where for (p,ω), $\omega \in T_p^*M$ is $\omega = \dot{x}_i \, dx^i_{(p)}$

Exterior powers $\Lambda^q T^*M = \bigcup_{p\in M} \Lambda^q T_p^*M$, q-forms at p.

Smooth map $f: M \to N$ induced derivative $f_*: T_pM \to T_{f(p)}N$, $\forall p \in M$.

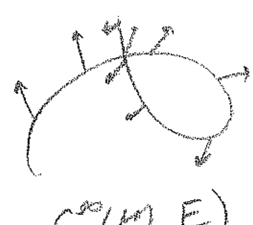
& its dual $f^*: T^*_{f(p)}N \to T_p^*M$.

Cross-sections of a vector bundle $E \to M$ is $C^\infty(M, E)$

Vector fields on M are elements of $C^\infty(M, TM)$

If $f: M \to X$ is immersed submanifold, then $C^\infty(M, TX)$

are vector fields along M.



Normal bundle $E = \bigcup_{p\in M} E_p$, where

$E_p = T_{f(p)}X / f_* T_pM$. If $v \in C^\infty(M, TX)$, let $[v] \in C^\infty(M, E)$ normal field

For brevity write $A^q(M) = C^\infty(M, \Lambda^q T^*M)$, smooth q-forms.

Smooth $f: M \to X$ induces $f^*: A^q(X) \to A^q(M)$

Exterior differential $d: A^q(M) \to A^{q+1}(M)$ commutes with f^*.

$$d(a_{i_1\cdots i_q}\, dx^{i_1}\wedge\cdots\wedge dx^{i_q}) = \frac{\partial a_{i_1\cdots i_q}}{\partial x^j}\, dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_q}.$$

$$f^*(d\alpha) = d(f^*\alpha), \text{ so } f^*(dx^i) = d(f^*x^i) = d(x^i\circ f)$$

<u>Canonical</u> 1-form Θ on T^*M $\quad (\Theta \in C^\infty(T^*M, T^*T^*M))$

If $v \in T_{(p,\omega)}T^*M$ is a tangent vector at $(p,\omega)\in T^*M$ and

$\pi: T^*M \to M$, $\pi(p,\omega) = p$ is the <u>projection</u>, then

$\Theta_{(p,\omega)}(v) = \omega(\pi_* v)$, where $\pi_* v \in T_pM$ & $\omega \in T_p^*M$.

In local coords $(x^i, \dot{x}_i)$, $\Theta = \dot{x}_i\, dx^i$ & $d\Theta = d\dot{x}_i \wedge dx^i$
<u>symplectic form</u> on T^*M.

Namely, $d\Theta$ is a 2-form & $d\Theta\wedge\cdots\wedge d\Theta = (d\Theta)^n \neq 0$ at every point.

<u>Flow</u> of a vector field v on M is a 1-parameter pseudo-group of local diffeos $\{\gamma_t\}$ on M. Fix $p\in M$, $t \mapsto \gamma_t(p)$ is the integral curve of v with $\gamma_0(p) = p$.

That is, $\dot{\gamma}_t(p) = v(\gamma_t(p))$ (velocity vector).

Lie derivative of $\varphi \in A^q(M)$ by $v \in C^\infty(M, TM)$ is

$$(L_v\varphi)_p = \frac{d}{dt}\Big|_0 \gamma_t^*\varphi = \lim_{t\to 0} \frac{\gamma_t^*\varphi_{\gamma_t(p)} - \varphi_p}{t} \text{ in } \Lambda^q T_p^*M.$$

$L_v: A^*(M) \to A^*(M)$ is a derivation of degree 0. On fns $f \in A^0(M)$, $L_v f = v(f)$

Henri Cartan's Formula

$$L_v \varphi = v \lrcorner d\varphi + d(v \lrcorner \varphi)$$

where *interior product* (contraction) "hook" is defined

pointwise: for $v \in V$, $v \lrcorner : \Lambda^q V^* \to \Lambda^{q-1} V^*$

$$(v \lrcorner \psi)(u_2, \ldots, u_q) = \psi(v, u_2, \ldots, u_q) \quad u_i \in V$$

$v \lrcorner : \Lambda T_p^* M \to \Lambda T_p^* M$ is an anti-derivation, i.e. derivation of degree -1.

If $\alpha \in T_p^* M$, then $v \lrcorner \alpha = \alpha(v)$

$\therefore$ If $\alpha, \beta \in T_p^* M$, $c \in \mathbb{R}$, then $v \lrcorner (c\, \alpha \wedge \beta) = c(\alpha(v)\beta - \beta(v)\alpha)$.

Exercise If V, W are finite dimensional vector spaces over $\mathbb{R}$, with dual spaces V^* and W^*, resp., and if $T: V \to W$ is a linear transf. with dual $T^*: W^* \to V^*$, then for any $v \in V$ and $\theta \in W^*$,

$$v \lrcorner (T^* \theta) = (Tv) \lrcorner \theta. \qquad (\in \mathbb{R}).$$

More generally, T & T^* induce linear maps $T: \Lambda V \to \Lambda W$ & $T^*: \Lambda W^* \to \Lambda V^*$, and for any $v \in V$, $\varphi \in \Lambda^q W^*$,

$$v \lrcorner T^* \varphi = T^*((Tv) \lrcorner \varphi) \qquad \in \Lambda_{q-1} V^*$$

most elegant solution: define $v \lrcorner$ as dual to multiplication by v, $\varepsilon(v): \Lambda V \to \Lambda V$ and isom. $(\Lambda V)^* \cong \Lambda V^*$ given by pairing $\Lambda V \times \Lambda V^* \to \mathbb{R}$

$$\langle v_1 \wedge \cdots \wedge v_r, \theta^1 \wedge \cdots \wedge \theta^r \rangle = \det(\theta^i(v_j))$$

Then $\langle v \lrcorner T^* \varphi, x \rangle = \langle T^* \varphi, \varepsilon(v) x \rangle = \langle \varphi, T(\varepsilon(v) x) \rangle = \langle \varphi, \varepsilon(Tv) Tx \rangle$

$= \langle (Tv) \lrcorner \varphi, Tx \rangle = \langle T^*((Tv) \lrcorner \varphi), x \rangle, \quad \forall v \in V, \ \varphi \in \Lambda V^*, \ x \in \Lambda V.$

Frame bundle & Cartan-Darboux.

Bundle of oriented o.n. frames $\mathcal{F}(\mathbb{R}^3) \cong$ Euclidean group of proper motions $E(3) = \left\{ \begin{vmatrix} 1 & 0 \\ x & A \end{vmatrix} : \begin{matrix} x \in \mathbb{R}^3 \\ A \in SO(3) \end{matrix} \right\}$ ($= (x, A)$) $\subseteq GL(4,\mathbb{R})$.

Group element $(x, A) \leftrightarrow$ frame e_1, e_2, e_3 at x, where $e_i =$ col i of A.

Maurer-Cartan form $(\omega^i, \omega^i_j) = (x, A)^{-1} d(x, A) = (A^{-1}dx, A^{-1}dA)$.

That is $dx = \omega^i e_i$ so $\omega^i = dx \cdot e_i$ 1-forms

$de_i = \omega^j_i e_j$, $\omega^j_i = de_i \cdot e_j = -e_i \cdot de_j = -\omega^i_j$, $i, j = 1, 2, 3$.

Structure equations $d(\omega^i, \omega^i_j) = -(x, A)^{-1} d(x,A) (x, A^{-1}) \wedge d(x, A) = -((\omega^i), (\omega^i_j)) \wedge ((\omega^i), (\omega^i_j))$ entry

$= (-\omega^i_j \wedge \omega^j, -\omega^i_k \wedge \omega^k_j)$

in detail is $d\omega^i = -\omega^i_j \wedge \omega^j$

$d\omega^i_j = -\omega^i_k \wedge \omega^k_j$.

An o.n. frame field along a regular curve $x : N \to \mathbb{R}^3$, $s \mapsto x(s)$

is a lift $(x, e) : N \to E(3)$, so $\forall s \in N$, $e_1(s), e_2(s), e_3(s)$ is an o.n. frame at $x(s)$.

First order if $\dot{x}(s) = \lambda e_1(s)$, $\lambda(s) > 0$.

s is arclength parameter means $|\dot{x}(s)| = 1$ so $\lambda = 1$, & $\dot{x} = e_1$.

Then $dx = \omega^1 e_1$, so $\omega^2 = \omega^3 = 0$ characterizes first order frame fields

Here ω^i means $x^* \omega^i$, pull-back to N.

s arc-length par. $\Leftrightarrow x^*\omega^1 = ds$.

Second order if $\ddot{x}(s) = \dot{e}_1(s) = \kappa(s) e_2(s)$, then κ called <u>curvature</u>.

$\ddot{x} \equiv 0 \Rightarrow \kappa$ not defined

Now $e_3 = e_1 \times e_2$ is already determined

$de_1 = \dot{e}_1 ds = \kappa(s) ds\, e_2(s)$ so second order $\Rightarrow x^* \omega^3_1 = 0$

$x^* \omega^2_1 = \kappa(s) \omega^1 = \kappa(s) ds$.

$\overset{\shortparallel}{\omega^2_1} e_2 + \omega^3_1 e_3$

Then $\omega^1_2 e_1 + \omega^3_2 e_3 = de_2 = -\kappa \omega^1 e_1 + \omega^3_2 e_3$ so $\omega^3_2 = \tau \omega^1$, $\tau =$ torsion.

2nd order is Frenet frame. Frenet-Serret eqns

$$\frac{dx}{ds} = e_1$$

$$\frac{de_1}{ds} = \kappa(s) e_2$$

$$\frac{de_2}{ds} = -\kappa(s) e_1 + \tau(s) e_3$$

$$\frac{de_3}{ds} = -\tau(s) e_2$$

i.e. $(x, e)^{-1} d(x, e) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \right) ds$

Cartan-Darboux Theorem:

G Lie subgroup of $GL(n,\mathbb{R})$.

$\mathfrak{g} \subseteq \mathfrak{gl}(n,\mathbb{R})$ its Lie algebra of left-invariant vector fields.

$\omega_{(A)} = A^{-1}dA$ is $\mathfrak{g}$-valued Maurer-Cartan form.

$d\omega = -\omega\wedge\omega$ structure equations

$\omega = (\omega^i_j)$, $1\le i,j\le n$, $d\omega^i_j = -\omega^i_k\wedge\omega^k_j$.

(In general $\mathfrak{g}\subseteq\mathfrak{gl}(n,\mathbb{R})$ defined by a set of linear equations in ω^i_j).

1. Uniqueness. If M is a connected manifold, then smooth maps $f, \tilde f : M \to G$ are congruent (i.e. $\exists\, A\in G$ such that $\tilde f(x) = A f(x)$ $\forall\, x\in M$) iff

$$\tilde f^*\omega = f^*\omega$$

Remark. If $\eta = f^*\omega$, a $\mathfrak{g}$-valued 1-form on M, then

$$d\eta = d\underbrace{f^*\omega}_{f(x)^{-1}df(x)} = f^*d\omega = f^*(-\omega\wedge\omega) = -\eta\wedge\eta.$$

2. Existence Suppose η is a $\mathfrak{g}$-valued 1-form on a simply connected manifold M. Then there exists a map $f: M\to G$ such that $f^*\omega = \eta$ iff $d\eta = -\eta\wedge\eta$.

Proofs. Uniqueness. If $\tilde f = Af$, then $\tilde f^*\omega = f^* L_A^*\omega = f^*\omega$, since $L_A^*\omega = \omega$. Conversely, if $\tilde f^*\omega = f^*\omega$, then $F: M\to G$, $F(x) = \tilde f(x) f(x)^{-1}$ satisfies

$$dF_{(x)} = d\tilde f(x)\, f(x)^{-1} - \tilde f(x) f(x)^{-1} df(x) f(x)^{-1} = \tilde f(x)\underbrace{\left(\tilde f(x)^{-1}d\tilde f(x) - f(x)^{-1}df_{(x)}\right)}_{\tilde f^*\omega - f^*\omega = 0} f(x)^{-1}$$

$\Rightarrow F$ is constant on M, say $F(x) = A$, $\forall\, x\in M$.

$\therefore\ A = \tilde f(x) f(x)^{-1}$ $\forall\, x\in M$.

<u>Existence</u> An application of the Frobenius Theorem gives local existence. Global existence on a simply connected M is a more complex argument. Ref. Sarfbook.

<u>Example</u>

<u>Uniqueness</u>: $f, \tilde{f} : \overset{\text{interval}}{N} \to \mathbb{R}^3$ have Frenet frames (f, e) & $(\tilde{f}, \tilde{e})$ with same curvatures & torsions, $\kappa(s) = \tilde{\kappa}(s)$, $\tau(s) = \tilde{\tau}(s)$, $\forall s \in N$.

Then $\exists$ a rigid motion $(\vec{v}, A) \in \mathbb{E}(3)$ such that

$$\tilde{f}(s) = \vec{v} + A f(s), \quad \forall s \in N;$$

i.e. they're <u>congruent</u>.

Proof. $$(\tilde{f}, \tilde{e})^{-1} d(\tilde{f}, \tilde{e}) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & -\tilde{\kappa} & 0 \\ +\tilde{\kappa} & 0 & -\tilde{\tau} \\ 0 & +\tilde{\tau} & 0 \end{pmatrix} \right) ds = (f, e)^{-1} d(f, e) \quad \forall s \in N$$

<u>Existence</u>. Let $N \subset \mathbb{R}$ be an interval & let $\kappa(s)$, $\tau(s)$ be any smooth (class C^1 is enough?) functions on N. Then there exists a smooth map $(f, e) : N \to \mathbb{E}(3)$ such that

$$\eta = (f, e)^{-1} d(f, e) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & -\kappa & 0 \\ +\kappa & 0 & -\tau \\ 0 & +\tau & 0 \end{pmatrix} \right) ds$$

In fact, the integrability condition on η, $d\eta = -\eta \wedge \eta$, is vacuous for dimension reasons here.

<u>Exercise</u> 1). A diffeomorphism $F: M \to M$ induces a bundle map
$$F^*: T^*M \to T^*M \quad \text{given by} \quad F^*(p,\alpha) = (F^{-1}(p), F^*\alpha)$$
(bundle map means fibre preserving, linear on fibers.)

2. $$\begin{tikzcd} T^*M \arrow[r, "F^*"] \arrow[d, "\pi"'] & T^*M \arrow[d, "\pi"] \\ M & M \arrow[l, "F"] \end{tikzcd}$$
satisfies $\pi \circ F^* = F^{-1} \circ \pi$, so $d\pi \circ dF^* = dF^{-1} \circ d\pi$

3. If Θ is the canonical 1-form on T^*M, so $\Theta_{(p,\alpha)}(v) = \alpha(d\pi\, v)$,

then $F^*\Theta = \Theta$.

[Soln: If $v \in T_{(p,\alpha)} T^*X$, then

$$\left((F^*)^*\Theta\right)\Big|_{(p,\alpha)}(v) = \Theta_{F^*(p,\alpha)}(dF^* v) = \Theta_{(F^{-1}(p),\, F^*\alpha)}(dF^* v) = (F^*\alpha)\big(\overbrace{d\pi \circ dF^* v}^{dF^{-1}\circ d\pi}\big)$$

$$= \alpha(dF \circ dF^{-1} \circ d\pi\, v) = \alpha(d\pi\, v) = \Theta_{(p,\alpha)}(v). \quad \lrcorner$$