2. Jet spaces

$J^k(\mathbb{R}, M)$ is the set of equivalence classes of maps $f: \mathbb{R} \to M$, where $f \sim \tilde{f}$ at $x \in \mathbb{R}$ if $f(x) = \tilde{f}(x)$ and in some (hence any) local coord $(y^1, \ldots, y^m)$ about $f(x)$, the Taylor series expansions of $f^\alpha = y^\alpha \circ f$ and $\tilde{f}^\alpha = y^\alpha \circ \tilde{f}$ at x agree through order k.

$j^k_x(f)$ denotes such an equivalence class: k-jet of f at x.

loc. coord x on $\mathbb{R}$ and (y^α), $\alpha = 1, \ldots, m = \dim M$ local coords in M induce local coords $(x, y^\alpha, \dot{y}^\alpha)$ on $J^1(\mathbb{R}, M)$ by

$(x, y^\alpha, \dot{y}^\alpha)\, j^1_x(f) = (x, y^\alpha \circ f(x), \frac{d}{dx}(y^\alpha \circ f)(x))$;

+ local coords $(x, y^\alpha, \dot{y}^\alpha, \ddot{y}^\alpha)$ on $J^2(\mathbb{R}, M)$ by

$(x, y^\alpha, \dot{y}^\alpha, \ddot{y}^\alpha)\, j^2_x(f) = (x, y^\alpha \circ f(x), \frac{d}{dx}(y^\alpha \circ f)(x), \frac{d^2}{dx^2}(y^\alpha \circ f)(x))$

etc on $J^k(\mathbb{R}, M)$, $k = 1, 2, 3, \ldots$ $\quad \dim J^k(\mathbb{R}, M) = 1 + (k+1)m$

Note $J^1(\mathbb{R}, M) \cong \mathbb{R} \times TM$

$$j^1_x(f) \longleftrightarrow (x, (f(x), df(\tfrac{\partial}{\partial x})))$$

A class C^k map $f: \mathbb{R} \to M$ induces a continuous lift $j^k(f): \mathbb{R} \to J^k(\mathbb{R}, M)$ by $j^k(f)(x) = j^k_x(f)$.

<u>Exercise</u>. Suppose $y=(y^1,\dots,y^m)$ and $z=(z^1,\dots,z^m)$ are coordinate charts on $\mathcal{U}\subseteq M$. Then $z\circ y^{-1}: y(\mathcal{U})\subseteq\mathbb{R}^m \to z(\mathcal{U})\subseteq\mathbb{R}^m$ is a diffeo. $\forall p\in M$, $z^\alpha(p) = z^\alpha(y^{-1}\circ y(p)) = (z\circ y^{-1})^\alpha(y(p))$

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$z^\alpha(y^1,\dots,y^m)$ is the sense in which z^α is a fcn of y.

Then $dz^\alpha(p) = \frac{\partial z^\alpha}{\partial y^\beta}(p)\,dy^\beta(p)$

Prove that the coordinates $x, y^\alpha, \dot y^\alpha$ and $x, z^\alpha, \dot z^\alpha$ on $J^1(\mathbb{R},M)$ are related by $dz^\alpha = \frac{\partial z^\alpha}{\partial y^\beta}dy^\beta$, $d\dot z^\alpha = \dot y^\beta \frac{\partial^2 z^\alpha}{\partial y^\beta \partial y^\gamma}dy^\gamma + \frac{\partial z^\alpha}{\partial y^\beta}d\dot y^\beta$

If $\theta^\alpha = dy^\alpha - \dot y^\alpha dx$ and $\tilde\theta^\alpha = dz^\alpha - \dot z^\alpha dx$, then

$$\tilde\theta^\alpha = \frac{\partial z^\alpha}{\partial y^\beta}\theta^\beta$$

On $J^2(\mathbb{R},M)$, if $\dot{\tilde\theta}^\alpha = d\dot z^\alpha - \ddot z^\alpha dx$ and $\dot\theta^\alpha = d\dot y^\alpha - \ddot y^\alpha dx$, then

$$\dot{\tilde\theta}^\alpha = \frac{\partial z^\alpha}{\partial y^\beta}\dot\theta^\beta + \dot y^\beta\frac{\partial^2 z^\alpha}{\partial y^\beta\partial y^\gamma}\theta^\gamma .$$

Canonical subbundle $W^* \subset T^*(J^k(\mathbb{R},M))$ is

<u>$k=1$</u>. $W^* = \operatorname{span}\{\theta^1,\dots,\theta^m\}$, where $\theta^\alpha = dy^\alpha - \dot y^\alpha dx$ in local coords.

<u>$k=2$</u>. $W^* = \operatorname{span}\{\theta^1,\dots,\theta^m, \dot\theta^1,\dots,\dot\theta^m\}$, where $\dot\theta^\alpha = d\dot y^\alpha - \ddot y^\alpha dx$

By the Exercise, W^* does not depend on the choice of local coordinates $y^1,\dots,y^m$ in M.

<u>Differential ideal</u> is a graded ideal $\mathcal{I} = \bigoplus_{g \geq 0} \mathcal{I}^g \subset \underset{\substack{\uparrow \\ \text{algebra over } \mathbb{R} \\ \text{module over } C^\infty(X)}}{A^*(X)}$

of C^∞ diff forms on a man. X such that

$$d\mathcal{I} \subset \mathcal{I} \qquad (\text{so } d\mathcal{I}^g \subset \mathcal{I}^{g+1} \text{ etc})$$

Graded means: if $\varphi = \varphi^0 + \varphi^1 + \dots + \varphi^n \in \mathcal{I}$, $\varphi^k \in A^k(X)$,

then 1) $\varphi^k \in \mathcal{I}^k \subset \mathcal{I}$.

2) $\varphi \wedge \eta \in \mathcal{I}$, $\forall \eta \in A^*(X)$

3) $d\varphi \in \mathcal{I}$

Subset $\Sigma \subset A^*(X)$, let $\{\Sigma\}$ denote <u>algebraic ideal</u> gen. by Σ.

Diff ideal gen by Σ is $\{\Sigma, d\Sigma\}$, where $d\Sigma = \{d\varphi : \varphi \in \Sigma\}$.

(note: if $\varphi \in \Sigma$ and $\alpha \in A^*(X)$, then $\alpha \wedge d\varphi \in \{\Sigma, d\Sigma\}$ and

$d(\alpha \wedge d\varphi) = d\alpha \wedge d\varphi \in \{\Sigma, d\Sigma\}$ as well).

If Σ is a finite set of 1-forms $\theta^1, \dots, \theta^n$, then

$\mathcal{I} = \{\Sigma, d\Sigma\} = \{\theta^\alpha, d\theta^\alpha\}$ is a <u>Pfaffian diff. ideal</u>.

<u>Integral manifold</u> of $\mathcal{I}$ on X is a manifold N &

an immersion $f: N \to X$ s.t. $f^*\varphi = 0$, $\forall \varphi \in \mathcal{I}$.

Usually f will be an embedding, so we write $N \subset X$

is a submanifold & we write $f^*\varphi = \varphi_N$.

If $\mathcal{I} = \{\Sigma, d\Sigma\}$, then $f: N \to X$ is a int. man iff

$$f^*\varphi = 0 \quad \forall \varphi \in \Sigma, \text{ since}$$

$\forall \alpha \in A^*(X)$, $f^*(\alpha \wedge \varphi) = f^*\alpha \wedge \underbrace{f^*\varphi}_{0}$ & $f^*(\alpha \wedge d\varphi) = f^*\alpha \wedge \underbrace{df^*\varphi}_{0}$.

Canonical Pfaffian diff ideal on $J^k(\mathbb{R}, M) = X$.

is generated by sections of the canonical subbundle $W^* \subset T^* J^k(\mathbb{R}, M)$

On $J^1(\mathbb{R}, M)$, coords $y^1, \ldots, y^m$ on M, $\theta^\alpha = dy^\alpha - \dot{y}^\alpha dx$ sections of W^*

$\mathcal{I} = \{\theta^\alpha, d\theta^\alpha\}$.

On $J^2(\mathbb{R}, M)$, $\dot{\theta}^\alpha = d\dot{y}^\alpha - \ddot{y}^\alpha dx$, $\mathcal{I} = \{\theta^\alpha, \dot{\theta}^\alpha, d\theta^\alpha, d\dot{\theta}^\alpha\}$.

Etc.

For any smooth immersed curve $f : \overset{[a,b]\subset\mathbb{R}}{N} \to M$, the lift $j^k f : N \to J^k(\mathbb{R}, M)$

is an integral submanifold of $\mathcal{I}$.

Case $k=1$, $j^1 f : N \to J^1(\mathbb{R}, M)$, $(j^1 f)(x) = j^1_x f$ in local coords $y^1, \ldots, y^m$ on M

is $y^\alpha(j^1 f)(x) = y^\alpha \circ f(x)$, $\dot{y}^\alpha(j^1 f)(x) = (y^\alpha \circ f)'(x)$.

$\Rightarrow (j^1 f)^* \theta^\alpha = (j^1 f)^*(dy^\alpha - \dot{y}^\alpha dx) = d(y^\alpha \circ f)(x) - (y^\alpha \circ f)'(x)\, dx = 0$

Case $k=2$. Also $(j^2 f)^* \dot{\theta}^\alpha = (j^2 f)^*(d\dot{y}^\alpha - \ddot{y}^\alpha dx) = d(y^\alpha \circ f)'(x) - (y^\alpha \circ f)''(x)\, dx = 0$. Etc.

any integral manifold $f : N \to J^k(\mathbb{R}, M)$ for which $f^* dx \neq 0$ is given

in this way.