3. Exterior differential systems

Exterior differential system $(\mathcal{I}, \omega)$ on manifold X

is a differential ideal $\mathcal{I}$ of $A^*(X)$ together with

an n-form ω. Call n the number of independent variables in $(\mathcal{I}, \omega)$.

$n = 1$ in these lectures. Note ω defined only modulo $\mathcal{I}$.

An integral element of $(\mathcal{I}, \omega)$ is a pair (p, E),

where $p \in X$ and $E \subset T_pX$ is an n-dim'l subspace,

such that: i) $\varphi|E = 0 \quad \forall \varphi \in \mathcal{I}$

ii) $\omega|E \neq 0$.

Let $V(\mathcal{I}, \omega) \subset G_n(X)$ be the set of all integral elements of $(\mathcal{I}, \omega)$.

Here $G_n(X) = \bigcup_{p \in X} G_n(T_pX)$ is the Grassmann bundle of

tangent n-planes of X.

An integral manifold of $(\mathcal{I}, \omega)$ is a smooth immersion $f: N \to X$,

where N^n is connected n-manifold and $f_*(T_pN) \in V(\mathcal{I}, \omega)$, $\forall p \in N$.

Explicitly, $f^*\varphi = 0$, $\forall \varphi \in \mathcal{I}$ and $f^*\omega \neq 0$. (the independence condition)

Let $\mathcal{V}(\mathcal{I}, \omega)$ be the set of all integral manifolds of $(\mathcal{I}, \omega)$.

Most (maybe all) of our examples will be Pfaffian diff systems, so $\mathcal{I}$ is generated by a subbundle $W^* \subset T^*X$, and $n=1$, so the independence condition is given by another subbundle $W^* \subset L^* \subset T^*X$ such that L^*/W^* is a line bundle and $\omega \in C^\infty(X, L^*)$ defines a nowhere zero section of L^*/W^*.

<u>Example</u>. On $X = J^1(\mathbb{R}, M)$, let $\mathcal{I}$ be the canonical Pfaffian system generated by $W^* = \operatorname{span}\{\theta^1, .., \theta^m\}$, where in terms of local coords $(x, y^1, .., y^m, \dot{y}^1, .., \dot{y}^m)$ induced on $J^1(\mathbb{R}, M)$ by local coords $(x, y^1, .., y^m)$ on $\mathbb{R} \times M$, $\theta^\alpha = dy^\alpha - \dot{y}^\alpha dx$.

Let $L^* = \pi^* T^*(\mathbb{R} \times M)$, where $\pi : J^1(\mathbb{R}, M) \to \mathbb{R} \times M$
$$j^1_x f \longmapsto (x, f(x))$$

Then $\omega = dx$ is a section of L^* inducing a nowhere section of L^*/W^*.

$(\mathcal{I}, \omega)$ is the <u>canonical</u> EDS on $J^1(\mathbb{R}, M)$.

An integral manifold is given by an immersion $f : N \to X$ where $N = [a, b] \subset \mathbb{R}$ and $f^*\theta^\alpha = 0$, $\alpha = 1, .., m$, $f^*\omega \neq 0$.

So, $f^*dx \neq 0 \Rightarrow x \circ f$ is a local coordinate on N, and f is
$$f(x) = (x, y^\alpha(x), \dot{y}^\alpha(x)), \text{ so } 0 = f^*\theta^\alpha = dy^\alpha - \dot{y}^\alpha dx = \left(\frac{dy^\alpha}{dx} - \dot{y}^\alpha(x)\right) dx$$

$\Rightarrow \dot{y}^\alpha(x) = \dfrac{dy^\alpha}{dx}(x)$. Hence $f = j^1 F$, where $F : N \to M$, $F(x) = (y^1(x), .., y^m(x))$.

An E.D.S $(\mathcal{I}', \omega')$ on X' can be restricted to a submanifold $X \subset X'$ to an EDS $(\mathcal{I}, \omega)$ on X. (If $i : X \hookrightarrow X'$, then $\mathcal{I} = i^*\mathcal{I}'$ is a diff ideal in $A^*(X)$, and $\omega' = i^*\omega$). Call $(\mathcal{I}, \omega)$ the EDS $(\mathcal{I}', \omega')$ <u>with</u> <u>constraints</u>.

<u>Example</u>. Let $Y \subset J^1(\mathbb{R}, M)$ be a submanifold and let $(\mathcal{I}, \omega)$ be the restriction to Y of the canonical EDS on $J^1(\mathbb{R}, M)$. Locally, Y is defined by equations

$$g^\rho(x, y^\alpha, \dot{y}^\alpha) = 0 \quad , \quad \rho = 1, \ldots, r,$$

so the integral manifolds $f : N \to Y$ of $(\mathcal{I}, \omega)$ are 1-jets of curves $F(x) = (x, y^\alpha(x))$ in $\mathbb{R} \times M$ satisfying the ODE's

$$g^\rho(x, y^\alpha(x), \frac{dy^\alpha}{dx}(x)) = 0 \quad , \quad \rho = 1, \ldots, r$$