4. Variational Problems

A variational problem $(\mathcal{I},\omega,\varphi)$ on X is a Pfaffian E.D.S. $(\mathcal{I},\omega)$ on a manifold X, a smooth 1-form φ on X, and the functional $\Phi : \tilde{\mathcal{V}}(\mathcal{I},\omega) \to \mathbb{R}$

$$(N,f) \mapsto \int_N f^*\varphi$$

where $\tilde{\mathcal{V}}(\mathcal{I},\omega) \subset \mathcal{V}(\mathcal{I},\omega)$ is the set of integral manifolds $f : N \to X$ for which the integral is defined.

The problem is to find the extremals of Φ. This requires making sense out of the vague idea of a variation of Φ at $(N,f) \in \tilde{\mathcal{V}}(\mathcal{I},\omega)$ as a "normal vector field" v along $f : N \to X$, defining an element in the "tangent space" $T_{(N,f)}\tilde{\mathcal{V}}(\mathcal{I},\omega)$ such that the "derivative"

$$\delta\Phi_{(N,f)}(v) = 0 \qquad \forall \text{ such } v.$$

This leads to the variational or Euler-Lagrange equations of Φ.

In the classical development of [Gelfand-Fomin], at this point function spaces and functional analysis plays a large role. We shall proceed quite differently.

Example (Classical variational problems).

M^m a manifold, $X = J^1(\mathbb{R}, M)$, $(\mathcal{I}, \omega)$ the canonical EDS.

A Lagrangian is a function $\mathcal{L}: X \to \mathbb{R}$. If x is the coord. on $\mathbb{R}$, let $\varphi = \mathcal{L}\,dx$, a 1-form on X.

$(\mathcal{I}, \omega, \varphi)$ is a classical variational problem.

In terms of local coords $(x, y^1, \dots, y^m)$ on $\mathbb{R} \times M$, which induce $x, y^\alpha, \dot{y}^\alpha$ on X. Any $(N, f) \in \mathcal{V}(\mathcal{I}, \omega)$ is locally the 1-jet of a curve $F: N \to M$, $x \mapsto (y^\alpha(x))$; $\text{so } f(x) = (x, y^\alpha(x), \frac{dy^\alpha}{dx}(x))$

so $\quad \Phi(N, f) = \int_N \mathcal{L}(x, y^\alpha(x), \frac{dy^\alpha}{dx}(x))\,dx$

We postpone discussion of the class of functions $y^\alpha(x)$ intended: class C^1, C^∞? Endpoint conditions?

Example (Mechanical systems)

Let $T: TM \to \mathbb{R}$ be a smooth map such that for each $p \in M$, $T|T_pM \to \mathbb{R}$ is a positive definite quadratic form.

In local coordinates y^α on M, $\quad T = \frac{1}{2} g_{\alpha\beta}(y)\,\dot{y}^\alpha \dot{y}^\beta$

$(y^\alpha, \dot{y}^\alpha)$ are local coordinates on $TM \cong J^1(\mathbb{R}, M)$

$$(y^\alpha, \dot{y}^\alpha)(p, v) = (y^\alpha(p), dy^\alpha_p(v))$$

Let $U: M \to \mathbb{R}$ be a smooth function.

Then $\mathcal{L}: J^1(\mathbb{R}, M) \to \mathbb{R}$, $\mathcal{L} = T - U$ is a Lagrangian, given in local coords by $\mathcal{L}(x, y^\alpha, \dot{y}^\alpha) = \frac{1}{2} g_{\alpha\beta}(y)\,\dot{y}^\alpha \dot{y}^\beta - U(y)$

T = kinetic energy, U = potential energy.

Here $\Phi(N, f) = \int_N T(y^\alpha(x), \dot{y}^\alpha(x)) - U(y^\alpha(x))\, dx$ is the

action functional for the path $(y^\alpha(x))$ in M.

Example (Classical variational problem with constraints).

$J^1(\mathbb{R}, M)$ with canonical system, x coord in $\mathbb{R}$.

$X \subset J^1(\mathbb{R}, M)$ a submanifold such that $dx|_X$ never zero.

Local coords x, y^α on $\mathbb{R} \times M$, $(x, y^\alpha; \dot{y}^\alpha)$ on $J^1(\mathbb{R}, M)$,

X given by equations $g^\rho(x, y, \dot{y}) = 0$, $\rho = 1, \ldots, r$.

Lagrangian $\mathcal{L} : X \to \mathbb{R}$ (usually is restriction to X of $\mathcal{L} : J^1(\mathbb{R}, M) \to \mathbb{R}$)

$\varphi = \mathcal{L}\, dx$.

Now the functional $\Phi(N, f) = \int_N \mathcal{L}(x, y^\alpha(x), \frac{dy^\alpha}{dx}(x))\, dx$

is defined on curves $f(x) = (x, y^\alpha(x), \frac{dy^\alpha}{dx}(x))$ that satisfy the

constraints $\quad g^\rho(x, y^\alpha(x), \frac{dy^\alpha}{dx}(x)) = 0$, $\rho = 1, \ldots r$.

Holonomic constraints if $g^\rho = g^\rho(x, y^\alpha)$, does not depend on $\dot{y}^\alpha$

otherwise non-holonomic.

If $g^\rho = g^\rho(y)$, then f comes from a curve that lies in

the submanifold of M defined by $g^\rho(y) = 0$, $\rho = 1, \ldots, r$.

Variations: To calculate $\delta\Phi_{(N,f)} = \frac{d}{dt}\Big|_0 \Phi(N, f_t)$, we need to understand

variations (N, f_t) of integral submanifolds of $(\mathcal{I}, \omega)$.