## 5. Infinitesimal variations of $(N, f)$.

Pfaffian EDS $(\mathcal{I}, \omega)$ on $X$ generated by $W^* \subset L^* \subset T^*X$.

$f: N \to X$ a smooth immersion, $(N, f) \in \mathcal{V}(\mathcal{I}, \omega)$.

Let $\varepsilon > 0$ and let $[0, \varepsilon] = \{t \in \mathbb{R} : 0 \leq t \leq \varepsilon\}$.

A <u>variation</u> of $(N, f)$ is a smooth map $F: N \times [0, \varepsilon] \to X$ s.t. $f_t : N \to X$ defined by $f_t(x) = F(x, t)$ is an integral submanifold of $(\mathcal{I}, \omega)$, $\forall t \in [0, \varepsilon]$, and $f_0 = f$.

Its <u>associated infinitesimal variation</u> is the vector field $v$ along $f$, $v \in C^\infty(N, f^{-1}TX)$, defined by

$$v(x) = F_{*(x,0)} \frac{\partial}{\partial t}.$$

We want to characterize such $v$ in terms of $(\mathcal{I}, \omega)$.

The <u>normal bundle</u> of $f: N \to X$ is the vector bundle $E \to N$ defined by $E = f^{-1}(TX / f_* TN)$. $\therefore$ $E_x = T_{f(x)} X / f_* T_x N$.

Any vector field $v$ along $f$ induces a section $[v]$ of $E$.

Thus $[v] = [\tilde{v}]$ iff $v(x) - \tilde{v}(x) \in f_* T_x N$, $\forall x \in N$.

<u>Prop 1</u> If $v \in C^\infty(N, f^{-1}TX)$ is the infinitesimal variation associated to the variation $f_t : N \to X$ of $(N, f) \in \mathcal{V}(\mathcal{I}, \omega)$, then

$$(*) \quad f^*\big(v \lrcorner d\theta + d(v \lrcorner \theta)\big) = 0, \quad \forall \text{ sections } \theta \text{ of } W^*.$$

*Proof.* The proof follows from the following two calculations. Let $i: N \to N \times [0,\varepsilon]$ be $i(x) = (x,0)$, so $f = F \circ i : N \to X$.

1). For any 1-form $\theta$ on $X$,

$$i^*\left(L_{\frac{\partial}{\partial t}} F^*\theta\right) = f^*(v \lrcorner d\theta + d(v \lrcorner \theta))$$

2). If $\theta$ is a section of $W^*$, then

$$i^*\left(L_{\frac{\partial}{\partial t}} F^*\theta\right) = 0.$$

Proof of 1). By H. Cartan's formula, on $N \times [0,\varepsilon]$,

$$L_{\frac{\partial}{\partial t}} F^*\theta = d\left(\tfrac{\partial}{\partial t} \lrcorner F^*\theta\right) + \tfrac{\partial}{\partial t} \lrcorner \overbrace{dF^*\theta}^{F^*d\theta}$$

$$= dF^*\left(\left(F_*\tfrac{\partial}{\partial t}\right) \lrcorner \theta\right) + F^*\left(\left(F_*\tfrac{\partial}{\partial t}\right) \lrcorner d\theta\right) \quad \text{by } \underline{\text{Exercise Lect 1.}}$$

$$= F^*\left( d\left(\left(F_*\tfrac{\partial}{\partial t}\right) \lrcorner \theta\right) + F_*\tfrac{\partial}{\partial t} \lrcorner d\theta \right)$$

$$\therefore \quad i^*\left(L_{\frac{\partial}{\partial t}} F^*\theta\right) = \underbrace{i^* F^*}_{(F\circ i)^* = f^*}\left( d\left(\left(F_*\tfrac{\partial}{\partial t}\right) \lrcorner \theta\right) + F_*\tfrac{\partial}{\partial t} \lrcorner d\theta\right)$$

$$= f^*\left( d(v \lrcorner \theta) + v \lrcorner d\theta \right) \quad \because \left(F_*\tfrac{\partial}{\partial t}\right)_{i(x)} = v(x) \quad \forall x \in N.$$

Proof of 2) Suppose $\theta$ is a section of $W^*$. Then $f_t^*\theta = 0 = f_t^* d\theta$ for $0 \le t \le \varepsilon$, because $(N, f_t)$ is an integral submanifold of $(\vartheta, \omega)$.

$\therefore$ $F^*\theta = g(x,t)\,dt + h(x,t)\,dx = g(x,t)\,dt$, because $h(x,t)\,dx = f_t^*\theta$

$\therefore$ $L_{\frac{\partial}{\partial t}} F^*\theta = g_t(x,t)\,dt$ and $i^* L_{\frac{\partial}{\partial t}} F^*\theta = 0.$ //

Remarks. 1). An infinitesimal variation $v$ of a variation $f_t : N \to X$ of $(N, f) \in \mathcal{V}(\mathcal{L}, \omega)$ is characterized by ⊛ in Prop 1. We have not yet considered the question of whether a vector field $v \in C^\infty(N, f^{-1}TX)$ that satisfies ⊛ is associated to an actual variation $f_t : N \to X$. Nevertheless, as a first approx. we let the "tangent space at $(N,f)$" be

$$T_{(N,f)} \mathcal{V}(\mathcal{L}, \omega) = \{ v \in C^\infty(N, f^{-1}TX) : f^*(v \lrcorner d\theta + d(v \lrcorner \theta)) = 0 \}.$$

2). If $v \in C^\infty(N, f^{-1}TX)$ satisfies ⊛, and if $u$ is a vector field on $N$, then $\tilde{v} = v + f_* u$ also satisfies ⊛. In fact, if $\theta$ is any section of $W^*$, then

$$\tilde{v} \lrcorner d\theta = v \lrcorner d\theta + \underbrace{(f_* u) \lrcorner d\theta}_{\substack{\shortparallel \\ u \lrcorner f^* d\theta = 0, \ \because f^* d\theta = 0}}$$

$$d(\tilde{v} \lrcorner \theta) = d(v \lrcorner \theta + \underbrace{(f_* u) \lrcorner \theta}_{\substack{\shortparallel \\ u \lrcorner f^* \theta = 0}})$$

Hence, if any representative $v \in C^\infty(N, f^{-1}TX)$ of a normal vector $[v] \in C^\infty(N, TX / f_* TN)$ satisfies ⊛, then so also does any other representative.

The solution set of (*), the "set of variation vectors" at $(N,f) \subset \mathcal{V}(I,\omega)$ is the kernel of the first order linear differential operator

$$L : C^\infty(N,E) \to C^\infty(N, W \otimes T^*N) \quad , \text{ where}$$

$E = TX/_{TN}$ is the normal bundle of $f: N \to X$

and in terms of a frame field $\theta^\alpha$, $\alpha = 1,\ldots,s$ of $W^*$ and its dual frame $w_\alpha$ of $W \subset TX$,

$$L([v]) = w_\alpha \otimes f^*(v \lrcorner d\theta^\alpha + d(v \lrcorner \theta^\alpha)).$$

More detailed knowledge of Ker L requires more details about the Pfaffian EDS $(I,\omega)$. Postpone this for now (EDS in good form, the derived flag, Cauchy characteristics).