6. Euler-Lagrange equations

Variational problem $(\mathcal{I}, \omega, \varphi)$ on X.

For $(N,f) \in \mathcal{V}(\mathcal{I},\omega)$, $\quad \Phi(N,f) = \int_N f^*\varphi$.

$$T_{(N,f)}\mathcal{V}(\mathcal{I},\omega) = \{[v] \in C^\infty(N, TX/f_*TN) : (1)\ f^*(v \lrcorner d\theta + d(v \lrcorner \theta)) = 0,\ \forall \theta \in C^\infty(X, W^*)\}$$

a <u>variation</u> of $(N,f) \in \mathcal{V}(\mathcal{I},\omega)$ is a smooth map $F: N\times[0,\varepsilon] \to X$, for some $\varepsilon > 0$ such that if $i_t : N \to N\times[0,\varepsilon]$, $i_t(x) = (x,t)$, $\forall\, 0 \le t \le \varepsilon$, then $f_t = F \circ i_t : N \to X$ is an integral manifold of $(\mathcal{I},\omega)$.

Then $v(x) = F_{*(x,0)} \frac{\partial}{\partial t} : N \to TX$ & $[v] \in C^\infty(N, TX/f_*TN)$ is the <u>infinitesimal variation</u> of F. It satisfies (1) (Prop, Lecture 5).

Define $\delta\Phi_{(N,f)}([v]) = \frac{d}{dt}\Big|_0 f_t^*\varphi$

<u>Lemma 1</u> For any smooth map $F: N\times[0,\varepsilon] \to X$ and any 1-form φ on X,

$$\frac{d}{dt}\Big|_0 \int_N f_t^*\varphi = \int_N i_0^*(L_{\frac{\partial}{\partial t}} F^*\varphi) = \int_N f^*(v \lrcorner d\varphi + d(v \lrcorner \varphi))$$

Proof. $f_t^*\varphi = (F\circ i_t)^*\varphi = i_t^* F^*\varphi$.

$F^*\varphi = g(x,t)\,dx + h(x,t)\,dt$ is a smooth 1-form on $N\times[0,\varepsilon]$

$$f_t^*\varphi = i_t^*F^*\varphi = g(x,t)\,dx$$

$$L_{\frac{\partial}{\partial t}} F^*\varphi = \frac{\partial}{\partial t} \lrcorner dF^*\varphi + d\left(\frac{\partial}{\partial t} \lrcorner F^*\varphi\right) = \frac{\partial}{\partial t} \lrcorner \left(g_t\, dt\wedge dx + h_x\, dx \wedge dt\right) + dh$$

$$= g_t\,dx - \cancel{h_x\,dx} + \cancel{h_x\,dx} + h_t\,dt = g_t\,dx + h_t\,dt \Rightarrow i_0^*(L_{\frac{\partial}{\partial t}} F^*\varphi) = g_t(x,0)\,dx$$

$\because i_0^*dt = 0$

$$\therefore \frac{d}{dt}\Big|_0 \int_N f_t^*\varphi = \frac{d}{dt}\Big|_0 \int_N g(x,t)\,dx = \int_N g_t(x,0)\,dx.$$

& $\int_N i_0^*(L_{\frac{\partial}{\partial t}} F^*\varphi) = \int_N g_t(x,0)\,dx$. This proves the first equality.

2nd $i_0^*(L_{\frac{\partial}{\partial t}} F^*\varphi) = i_0^*\left(\frac{\partial}{\partial t} \lrcorner dF^*\varphi + d\left(\frac{\partial}{\partial t} \lrcorner F^*\varphi\right)\right) = i_0^*F^*\left(F_*\frac{\partial}{\partial t} \lrcorner d\varphi + d\left(\left(F_*\frac{\partial}{\partial t}\right) \lrcorner \varphi\right)\right)$

$= f^*(v \lrcorner d\varphi + d(v \lrcorner \varphi))$. //

Cor 1. In addition, if $v(a) = 0 = v(b)$, then

$$\delta \Phi_{(N,f)}([v]) = \int_N f^*(v \lrcorner d\varphi).$$

Proof. Now $\int_N f^* d(v \lrcorner \varphi) = \varphi_{f(b)}(v(b)) - \varphi_{f(a)}(v(a)) = 0 - 0 = 0$.

Extend our definition of the first variation of Φ as follows.

Let $T^0_{(N,f)} \mathcal{V}(\mathcal{d}, \omega) = \{ [v] \in T_{(N,f)} \mathcal{V}(\mathcal{d}, \omega) : v(a) = 0 = v(b) \}$.

Let $\delta\Phi_{(N,f)} \; T^0_{(N,f)} \mathcal{V}(\mathcal{d}, \omega) \to \mathbb{R}$, $\delta\Phi_{(N,f)}([v]) = \int_N f^*(v \lrcorner d\varphi)$.

Remark 1. If φ is replaced by $\tilde{\varphi} = \varphi + \Theta$, where Θ is any smooth section of $W^* \to X$, then $\tilde{\Phi}(N,f) = \int_N f^*\tilde{\varphi} = \int_N f^*\varphi = \Phi(N,f)$, since $f^*\Theta = 0$. The first variation of Φ is also unchanged, since $\forall v \in T^0_{(N,f)} \mathcal{V}(\mathcal{d}, \omega)$, $\int_N f^*(v \lrcorner d\tilde{\varphi}) = \int_N f^* v \lrcorner d\varphi + \underbrace{\int_N f^* v \lrcorner d\Theta}_{= f^* d(v \lrcorner \Theta)}$ by $(*)$

and $\int_N f^* d(v \lrcorner \Theta) = \Theta(v(b)) - \Theta(v(a)) = 0$.

Remark 2. If η is a section of the bundle $f^{-1} T^* X \to N$ (so $\eta(x) \in T^*_{f(x)} X$, $\forall x \in N$), then for any $v \in T^0_{(N,f)} \mathcal{V}(\mathcal{d}, \omega)$,

$$\int_N f^*(v \lrcorner d(\varphi + d\eta(v))) = \int_N f^*(v \lrcorner d\varphi), \quad \text{since } d\,d\eta(v) = 0,$$

and $\int_N f^* \varphi + d\eta(v) = \int_N f^*\varphi$, since

$$\int_N d\eta(v) = \eta_{f(b)}(v(b)) - \eta_{f(a)}(v(a)) = 0.$$

Prop If $(N,f) \in V(\partial,\omega)$ satisfies

$$\delta\Phi_{(N,f)}([v]) = \int_N f^*(v \lrcorner d\varphi) = 0 \quad \forall\, [v] \in T^\circ_{(N,f)} V(\partial,\omega),$$

then for some section Θ of W^*:

(2) $$f^*(u \lrcorner d(\varphi+\Theta)) = 0, \quad \forall\; u \in C^\infty(N, TX). \text{ (sections of } f^{-1}TX \to N)$$

These are the Euler-Lagrange equations of $(\partial,\omega,\varphi)$.

Proof. (Heuristic. Rigorous proof later).

Recall $[v] \in T^\circ_{(N,f)} V(\partial,\omega) \Rightarrow$ (1) $f^*(v \lrcorner d\Theta + d\, v \lrcorner \Theta) = 0$ $\forall$ sections Θ of W^*

and $v(a) = 0 = v(b)$.

(*) $f^*(v \lrcorner d\varphi) = d\,\eta(v)$, some function $\eta(v)$ (FTC or Poincaré lemma)

assumption 1: η is a smooth section of $f^{-1}T^*X \to N$.

$\therefore \int_N d\,\eta(v) = \eta_{f(b)}(v(b)) - \eta_{f(a)}(v(a)) = 0$ if $v(a) = 0 = v(b)$.

So $f^*(v \lrcorner d\varphi) = d\,\eta(v)$, whenever $v \in C^\infty(N, f^{-1}TX/TN)$

and $f^*(v \lrcorner d\Theta + d(v \lrcorner \Theta)) = 0$ $\forall$ sections Θ of W^*.

assumption 2 ($\exists$ a section Θ of W^* such that) (3) $f^*(u \lrcorner d\varphi) = -f^*((u \lrcorner d\Theta) - d(u \lrcorner \Theta)) + d\,\eta(u)$

for all sections $u \in C^\infty(N, f^{-1}TX)$.

Note this reduces to (*) if $u \in T^\circ_{(N,f)} V(\partial,\omega)$

It also reduces to (*) if $u \in C^\infty(N, f_*TN)$ is tangent to $N \subset X$.

assumption 3 (to eliminate derivatives of u in (3)):

$\eta : N \to f^{-1}T^*X$ is $\eta = \Theta \circ f$ i.e. $\eta(x) = \Theta_{f(x)} \in W^*_{f(x)} \subset T^*_{f(x)}X$.

where Θ is the section of W^* in assumption 2.

assumptions 1, 2, & 3 $\Rightarrow$: if $\delta\Phi_{(N,f)}[v] = 0 \ \forall\, [v] \in T^0_{(N,f)}\mathcal{V}(\mathcal{I},\omega)$,

then for all $u \in C^\infty(N, f^{-1}TX)$,

$$f^*(u \lrcorner d(\varphi+\Theta)) = f^*(u \lrcorner d\varphi) + f^*(u \lrcorner d\Theta)$$

$$= -f^*(u \lrcorner d\Theta + d(u \lrcorner \Theta)) + d(u \lrcorner \Theta) + f^*(u \lrcorner d\Theta) = 0.$$

Remark. The section Θ of W^* appearing in the Euler-Lagrange equations (2) will be determined by these equations. We shall go through several examples first.

Although the above used assumptions 1, 2, & 3, which have not been verified to hold in general, the following result is proved.

Cor. If $(N,f) \in \mathcal{V}(\mathcal{I},\omega)$ satisfies the Euler-Lagrange equations (2) for some section Θ of W^*, then its first variation is zero for any variation whose variation vector v satisfies $v(a) = 0 = v(b)$.

Proof. If v is the variation vector of a variation of (N,f) & $v(a)=0=v(b)$, then a fortiori, $v \in C^\infty(N, f^{-1}TN)$ so $f^*(v \lrcorner d(\varphi+\Theta)) = 0$ for some section Θ of W^*. $\therefore$ $0 = f^*(v \lrcorner d(\varphi+\Theta)) = f^*(v \lrcorner d\varphi) + \underbrace{f^*(v \lrcorner d\Theta)}_{= -f^* d(v \lrcorner \Theta)}$

$$\Rightarrow 0 = \int_N f^*(v \lrcorner d(\varphi+\Theta)) = \int_N f^*(v \lrcorner d\varphi) - \underbrace{\left(\Theta_{f(b)}(v(b)) - \Theta_{f(a)}(v(a))\right)}_{=0}.$$

Result now follows from Lemma 1.