7. The classical variational problems

1. A Classical variational problem $(\mathcal{I}, \omega, \varphi)$

$X = J^1(\mathbb{R}, M)$, with Lagrangian $\mathcal{L} : X \to \mathbb{R}$, so $\varphi = \mathcal{L}\,dx$,

$\omega = dx$, $\mathcal{I} = \{\theta^\alpha = dy^\alpha - \dot y^\alpha dx,\ d\theta^\alpha = -d\dot y^\alpha \wedge dx\}$ in terms of

a local coord chart $(y^\alpha) = (y^1, \ldots, y^m)$, on M, $m = \dim M$.

$\theta^1, \ldots, \theta^m$ is a frame field for $W^* \to X$, so any section of W^* is

$\Theta = \lambda_\alpha \theta^\alpha$, for some fcns $\lambda_\alpha : X \to \mathbb{R}$.

$$\therefore\ d(\varphi + \Theta) = \mathcal{L}_{y^\alpha} \underbrace{dy^\alpha \wedge dx}_{\theta^\alpha \wedge dx} + \mathcal{L}_{\dot y^\alpha}\, d\dot y^\alpha \wedge dx + d\lambda_\alpha \wedge \theta^\alpha - \lambda_\alpha\, d\dot y^\alpha \wedge dx$$

$$= (d\lambda_\alpha - \mathcal{L}_{y^\alpha} dx) \wedge \theta^\alpha + (\mathcal{L}_{\dot y^\alpha} - \lambda_\alpha)\, d\dot y^\alpha \wedge dx$$

The Euler-Lagrange equations for $(N, f) \in \mathcal{V}(\mathcal{I}, \omega)$ are:

$$f^*(u \lrcorner d(\varphi + \Theta)) = 0, \quad \forall\, u \in C^\infty(N, f^{-1}TX).$$

Write $u \lrcorner d(\varphi+\Theta) \equiv 0 \mod N$

To calculate these, we use the frame field of TX dual to

dx, θ^α, $d\dot y^\alpha$ of T^*X, which we write $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial \theta^\alpha}$, $\frac{\partial}{\partial \dot y^\alpha}$.

By Exercise 1 below, it suffices to consider only the cases $u = \frac{\partial}{\partial \theta^\alpha}$ and

$u = \frac{\partial}{\partial \dot y^\alpha}$.

$$\begin{cases} f^*\left(\frac{\partial}{\partial \dot y^\alpha} \lrcorner d(\varphi+\Theta)\right) = f^*\left(\frac{\partial \lambda_\alpha}{\partial \dot y^\alpha}\, \theta^\alpha + (\mathcal{L}_{\dot y^\alpha} - \lambda_\alpha)\, dx\right) = f^*(\mathcal{L}_{\dot y^\alpha} - \lambda_\alpha)\, dx = 0 \quad \because f^*\theta^\alpha = 0 \\ f^*\left(\frac{\partial}{\partial \theta^\alpha} \lrcorner d(\varphi+\Theta)\right) = f^*\left(\mathcal{L}_{y^\alpha} dx - d\lambda_\alpha\right) = 0 \end{cases}$$

Use mod(N) notation.

First eqn $\Rightarrow$ $\mathcal{L}_{\dot y^\alpha} - \lambda_\alpha = 0$ on N, since dx is never 0 on N.

(i.e. $(\mathcal{L}_{\dot y^\alpha} - \lambda_\alpha)\circ f = 0$ on N)

$\therefore$ second eqn $\Rightarrow$ $f^*(d\mathcal{L}_{\dot y^\alpha} - \mathcal{L}_{y^\alpha} dx) = 0$.

Any integral man. of $(\mathcal{J}, \omega)$ is the 1-jet of a curve $x \mapsto y(x)$ in M,

so $f(x) = (x, y^\alpha(x), \frac{dy^\alpha}{dx}(x))$, so $d\mathcal{L}_{\dot y^\alpha} = \frac{\partial \mathcal{L}_{\dot y^\alpha}}{\partial x} dx$,

and the second set of E-L-eqns are

$$(1). \qquad \frac{\partial \mathcal{L}_{\dot y^\alpha}}{\partial x} = \mathcal{L}_{y^\alpha} \qquad \text{(fns restricted to } N \text{ understood}$$
i.e. composed with f

These are the classical E-L eqns. (See Gelfand-Fomin p. 15).

2. A mechanical system.

Same $X = J^1(\mathbb{R}, M)$ and $(\mathcal{J}, \omega)$, but now the Lagrangian is

$\mathcal{L}(y, \dot y) = T(y, \dot y) - U(y)$, where

$U : M \to \mathbb{R}$ is the potential energy, and

$T(y, \dot y) = \frac{1}{2} g_{\alpha\beta}(y) \dot y^\alpha \dot y^\beta$ is the kinetic energy. ($g_{\alpha\beta}(y)$ pos. def $\forall y \in M$.)

$\therefore$ $\mathcal{L}_{\dot y^\alpha} = T_{\dot y^\alpha} = g_{\alpha\beta}(y) \dot y^\beta$ & $\mathcal{L}_{y^\alpha} = \frac{1}{2}\frac{\partial g_{\beta\gamma}(y)}{\partial y^\alpha} \dot y^\beta \dot y^\gamma - \frac{\partial U}{\partial y^\alpha}$

Then (1) become

$$(2) \qquad \frac{d}{dx}\left(g_{\alpha\beta}(y)\dot y^\beta\right) = \frac{1}{2}\left(\frac{\partial g_{\beta\gamma}}{\partial y^\alpha}(y(x)) \frac{dy^\beta}{dx} \frac{dy^\gamma}{dx}\right) - \frac{\partial U}{\partial y^\alpha}(y(x))$$

Let $(g^{\alpha\beta}(y))$ be the inverse matrix of $(g_{\alpha\beta}(y))$, so $\forall \alpha, \beta$

$$g^{\alpha\delta}(y)\, g_{\delta\beta}(y) = \delta^\alpha_\beta \,, \quad \text{and let}$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(\frac{\partial g_{\delta\beta}}{\partial y^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial y^\beta} - \frac{\partial g_{\beta\gamma}}{\partial y^\delta}\right) \qquad \text{(Christoffel symbols)}$$

Multiply (2) by $g^{\delta\alpha}$, sum on α, to get

$$(3) \quad \frac{d^2 y^\delta}{dx^2} + \Gamma^\delta_{\beta\gamma} \frac{dy^\beta}{dx}\frac{dy^\gamma}{dx} = -U^\delta \,, \quad \text{where } U^\delta = g^{\delta\alpha}\frac{\partial U}{\partial y^\alpha}.$$

In Riemannian geometry of the metric $g_{\alpha\beta}(y)\,dy^\alpha dy^\beta$ on M,

we have the curve γ given by $x \mapsto y(x)$, with velocity vector $\dot\gamma = \frac{dy^\alpha}{dx}\frac{\partial}{\partial y^\alpha}$

The covariant derivative of any vector field $v = v^\alpha(x)\frac{\partial}{\partial y^\alpha}$ is

$$\frac{Dv}{dx} = \left(\frac{dv^\alpha}{dx} + \Gamma^\alpha_{\beta\gamma}(y(x))\frac{dy^\beta}{dx} v^\gamma(x)\right)\frac{\partial}{\partial y^\alpha}.$$

The <u>acceleration</u> of γ is

$$\frac{D\dot\gamma}{dx} = \left(\frac{d^2 y^\alpha}{dx^2} + \Gamma^\alpha_{\beta\gamma}(y(x))\frac{dy^\beta}{dx}\frac{dy^\gamma}{dx}\right)\frac{\partial}{\partial y^\alpha}$$

The <u>gradient</u> of $U(y)$ is the vector field on M

$$\nabla U(y) = g^{\alpha\beta}(y)\frac{\partial U}{\partial y^\beta}\frac{\partial}{\partial y^\alpha}$$

Hence, the E-L equations (3) are

$$\frac{D\dot\gamma}{dx}(x) + \nabla U(y(x)) = 0,$$

the equations of motion relative to the force field $F = -\nabla U$.

<u>Remark</u>. Natural boundary conditions for the classical variational problem are fixed endpoints $\overset{\text{i.e.}}{\cancel{\text{of}}}$ the variation of the integral man. (N, f),

i.e. $y(a) = A$, $y(b) = B$, for some fixed points $A, B \in M$,

where $f(x) = (x, y(x), \frac{dy}{dx}(x))$

Then infinitesimal variations $v \in T_{(N,f)} \mathcal{V}(\mathcal{I}, \omega)$ would satisfy

$$\overset{\shortparallel}{v^0 \frac{\partial}{\partial x} + v^\alpha \frac{\partial}{\partial \theta^\alpha} + \dot{v}^\alpha \frac{\partial}{\partial \dot{y}^\alpha}} \quad \text{so } v^0 = v \lrcorner dx,\ v^\alpha = v \lrcorner \theta^\alpha$$
$$\dot{v}^\alpha = v \lrcorner d\dot{y}^\alpha.$$

$$v^0(a) = v^\alpha(a) = 0 = v^0(b) = v^\alpha(b).$$

<u>Exercise</u> Let V be a finite dimensional vector space $/\mathbb{R}$.
Let $\Psi \in \Lambda^2 V^*$ be an alternating bilinear form on V.
Prove that if $w, v_1, \ldots, v_m$ is any basis of V such that

$$\langle v_\alpha \lrcorner \Psi, w \rangle = 0 \quad , \text{ for } \alpha = 1, \ldots, m,$$

then $\langle v \lrcorner \Psi, w \rangle = 0$ for all $v \in V$.

[Pf $v = tw + t^\alpha v_\alpha$, $t, t^\alpha \in \mathbb{R}$

$$\langle v \lrcorner \Psi, w \rangle = \langle t w \lrcorner \Psi + t^\alpha v_\alpha \lrcorner \Psi, w \rangle = t \overbrace{\langle w \lrcorner \Psi, w \rangle}^{\langle \Psi, w \wedge w \rangle = 0} + t^\alpha \overbrace{\langle v_\alpha \lrcorner \Psi, w \rangle}^{0} = 0.]$$