8. Riemannian geometry and geodesics

Illustrate our methods in a coordinate free setting.
As an exercise one could set it up in $J^2(\mathbb{R}, S) \times \mathbb{R}$.

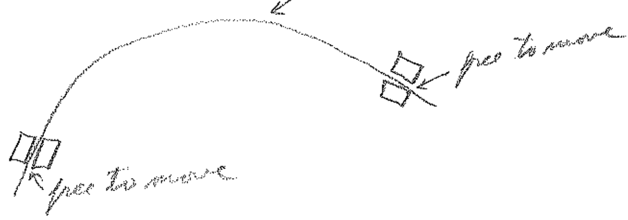
$S, \langle,\rangle$ a surface S with Riemannian metric $\langle,\rangle$.

$\gamma(s)$ a curve parametrized by arclength s, so $|\dot\gamma(s)| = 1 \ \forall s$,
and geodesic curvature $\kappa(s)$ (defined below).

Functional $\Phi(\gamma) = \frac{1}{2}\int_\gamma \kappa^2(s)\,ds$, for the elastic curve problem.

Investigate the Euler-Lagrange equations of Φ.

In $\mathbb{R}^2$, flat metric,



Riemannian geometry background:

$\pi: \mathcal{F}(S) \to S$ is the $O(2)$-bundle of o.n. frames on S.

Fiber over $p \in S$ is $\mathcal{F}(S)_p = \{(p, e_1, e_2) : p \in S,\ e_1, e_2$ o.n. basis of $T_pS\}$.

$O(2)$ acts on $\mathcal{F}(S)$ freely from the right:

$$(p, e_1, e_2)A = (p, (e_1, e_2)A).$$

The fiber $\mathcal{F}(S)_p$ is the $O(2)$-orbit through any point in it.

Projection $\pi(p, e_1, e_2) = p$.

Canonical forms ω^1, ω^2 on $\mathcal{F}(S)$ are smooth 1-forms defined by:

$\forall\, v \in T_{(p, e_1, e_2)}\mathcal{F}(S)$, $\pi_* v = \omega^1(v) e_1 + \omega^2(v) e_2$, so $\omega^i(v) = \langle \pi_* v, e_i \rangle$.

ω^1, ω^2 completes to a coframe field $\omega^1, \omega^2, \omega^2_1$ on $\mathcal{F}(S)$ by
the Levi-Civita connection form $\omega^2_1 = -\omega^1_2$ characterized by
the structure equations

$d\omega^1 = -\omega^1_2 \wedge \omega^2 = \omega^2_1 \wedge \omega^2$

$d\omega^2 = -\omega^2_1 \wedge \omega^1$

$d\omega^2_1 = -R\,\omega^1 \wedge \omega^2$, where $R = K \circ \pi$, K = Gaussian curvature on M.

Parametrized curve $\rho : N \to S$, $N \subset \mathbb{R}$, *immersion*.

An o.n. frame field along ρ is a smooth lift

$F : N \to \mathcal{F}(S)$, where *lift* means $\pi \circ F = \rho$

Thus, $\forall\, x \in N$, $F(x) = (\rho(x), e_1(x), e_2(x))$, where $e_1(x), e_2(x)$ is an

o.n. frame of $T_{\rho(x)} S$.

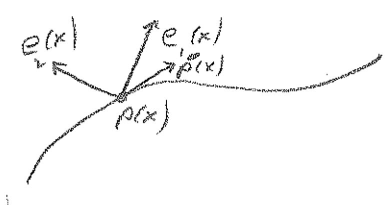
Let $F^*\omega^i = a^i dx$, $i = 1, 2$, $F^*\omega^2_1 = a^2_1 dx$, $a^1_2 = -a^2_1$ fns on N.

By definition of ω^1, ω^2, we have

$$\rho_* \frac{d}{dx} = \rho'(x) = \pi_* F_* \frac{d}{dx} = \pi_* F'(x) , \quad \text{where } F'(x) \in T_{F(x)} \mathcal{F}(S)$$

$$= \omega^1(F'(x))\, e_1(x) + \omega^2(F'(x))\, e_2(x)$$

$$= (F^*\omega^1)\left(\tfrac{d}{dx}\right) e_1(x) + (F^*\omega^2)\left(\tfrac{d}{dx}\right) e_2(x) = a^1(x)\, e_1(x) + a^2(x)\, e_2(x).$$



<u>Covariant derivative</u> of a vector field

$v(x) = v^1(x)\, e_1(x) + v^2(x)\, e_2(x)$ along ρ

is $$\frac{Dv}{dx} = \frac{dv^1}{dx} e_1 + v^1 (F^*\omega^2_1)(\rho')\, e_2 + \frac{dv^2}{dx} e_2 + v^2 (F^*\omega^1_2)(\rho')\, e_1$$

$$= \left(\frac{dv^1}{dx} + v^2 a^1_2\right) e_1 + \left(\frac{dv^2}{dx} + v^1 a^2_1\right) e_2$$

That is, $\dfrac{De_1}{dx} = \underbrace{(F^*\omega^2_1)(\rho')}_{=a^2_1}$ & $\dfrac{De_2}{dx} = \underbrace{(F^*\omega^1_2)(\rho')}_{=a^1_2}$ (compare to Christoffel symbols)

<u>Exercise</u> The vector field $\frac{Dv}{dx}$ is independent of the choice of lift $F = (\rho, e_1, e_2)$ along ρ.

Geometrically clear that we can choose the lift F so that

$F^*\omega^2 = 0$ (i.e. e_1 is tangent to ρ at each point).

Then $F^*\omega^1 = a^1 dx$ is never zero so $\exists$ fcn $s: N \to \mathbb{R}$ such that

$F^*\omega^1 = ds$. Call s an arclength parameter of the curve.

$= \int a(x)\,dx$

Now $\dot\rho(s) = \rho_* \frac{d}{ds} = e_1(s)$, and

$\ddot\rho(s) \overset{\text{def}}{=} \frac{D\dot\rho}{ds} = \frac{De_1}{ds} = \omega_1^2(\dot\rho) \overset{\text{def}}{=} \kappa(s)$, the geodesic curvature of ρ.

The curve is called a geodesic if $\kappa \equiv 0$ i.e. $\ddot\rho \equiv 0$.

<u>a variational problem</u>:

$X = \mathcal{F}(S) \times \mathbb{R}$, where $\mathbb{R}$ has coordinate κ

and $\mathcal{F}(S)$ has coframe field $\omega^1, \omega^2, \omega_1^2$.

Let $(\mathcal{I}, \omega)$ be $\begin{cases} \theta^1 = \omega^2 \\ \theta^2 = \omega_1^2 - \kappa\omega \\ \omega = \omega^1 \end{cases}$ so $d\theta^1 = -\omega_1^2 \wedge \omega^1 = -\theta^2 \wedge \omega$

$d\theta^2 = -R\omega^1\wedge\omega^2 - d\kappa\wedge\omega + \kappa\,\omega_2^1\wedge\omega^2$

$= R\,\theta^1\wedge\omega - d\kappa\wedge\omega - \kappa(\theta^2 + \kappa\omega)\wedge\theta^1$

$= (R+\kappa^2)\,\theta^1\wedge\omega - d\kappa\wedge\omega + \kappa\,\theta^1\wedge\theta^2$

$d\omega = -\omega_2^1 \wedge \omega^2 = (\theta^2 + \kappa\omega)\wedge\theta^1$

To warm up for the elastic curve, consider first the variational problem $(\mathcal{I}, \omega, \varphi)$, where $\varphi = \omega$, so $\Phi(N, f) = \int_N f^*\omega$ is the arc-length fnl.

for any integral manifold $f: N \to X$ of $(\mathcal{I}, \omega)$; that is

$f(s) = (\rho(s), e_1(s), e_2(s), \kappa(s))$ satisfies $f(s) = (F(s), \kappa(s))$, $F: N \to \mathcal{F}(S)$ is a frame field along $\rho(s)$

$0 = f^*\theta^1 = f^*\omega^2$,

$0 = f^*\theta^2 = f^*\omega_1^2 - \kappa(s) f^*\omega$

and the parameter s on N is chosen so that $f^*\omega = ds$.

$\therefore$ $\rho: N \to S$ satisfies $\dot\rho(s) = (f^*\omega^1)e_1 + (f^*\omega^2)e_2 = e_1$ & $f^*\omega_1^2 = \kappa(s)\,ds$

$\Rightarrow \kappa(s)$ is geodesic curv. of ρ.

Here $W^* = \mathrm{span}\{\theta^1, \theta^2\}$, so an arbitrary section is

$$\theta = \lambda_1\theta^1 + \lambda_2\theta^2, \quad \lambda_1, \lambda_2 \text{ functions on } X.$$

To calculate the Euler-Lagrange equations at $(N, f) \in \mathcal{V}(\mathcal{I}, \omega)$:

$$d(\varphi + \lambda_1\theta^1 + \lambda_2\theta^2) = d\omega + d\lambda_1\wedge\theta^1 + d\lambda_2\wedge\theta^2 + \lambda_1 d\theta^1 + \lambda_2 d\theta^2$$

$$= \kappa\,\omega\wedge\theta^1 - \theta^1\wedge\theta^2 + d\lambda_1\wedge\theta^1 + d\lambda_2\wedge\theta^2 + \lambda_1(-\theta^2\wedge\omega) + \lambda_2\left[(R+\kappa^2)\theta^1\wedge\omega - d\kappa\wedge\omega + \kappa\,\theta^1\wedge\theta^2\right]$$

$$= \left[d\lambda_1 + (\kappa - \lambda_2(R+\kappa^2))\omega\right]\wedge\theta^1 + \left[d\lambda_2 + \lambda_1\omega\right]\wedge\theta^2 - \lambda_2\, d\kappa\wedge\omega$$

$$+ (\lambda_2\kappa - 1)\,\theta^1\wedge\theta^2$$

a frame field dual to $d\kappa, \theta^2, \theta^1, \omega$ on X is

$$\frac{\partial}{\partial\kappa}, \ \frac{\partial}{\partial\theta^2}, \ \frac{\partial}{\partial\theta^1}, \ \frac{\partial}{\partial\omega}.$$

1) $f^*\left(\frac{\partial}{\partial\kappa} \lrcorner\, d(\varphi + \lambda_1\theta^1 + \lambda_2\theta^2)\right) = -\lambda_2\omega$

2) $f^*\left(\frac{\partial}{\partial\theta^2} \lrcorner\, d(\varphi + \lambda_1\theta^1 + \lambda_2\theta^2)\right) = -(d\lambda_2 + \lambda_1\omega)$

3) $f^*\left(\frac{\partial}{\partial\theta^1} \lrcorner\, d(\varphi + \lambda_1\theta^1 + \lambda_2\theta^2)\right) = -\left(d\lambda_1 + (\kappa - \lambda_2(R+\kappa^2))\omega\right)$

By the Exercise, if these are zero, then $\frac{\partial}{\partial\omega} \lrcorner (\quad) = 0$ also.

Important note: $f^*\theta^1 = 0 = f^*\theta^2 \Rightarrow$ that we need to calculate $d(\varphi + \theta)$ only modulo $\theta^1\wedge\theta^2$.

This also justifies our claim that, for example, $f^*\left(\overbrace{\frac{\partial}{\partial\theta^2} \lrcorner (d\lambda_1\wedge\theta^1)}^{(\frac{\partial}{\partial\theta^2} \lrcorner\, d\lambda_1)\theta^1}\right) = 0$

Since $f^*\omega$ is never 0 on N,

1) $\Rightarrow \lambda_2 = 0$
2) $\Rightarrow \lambda_1 = 0$
3) $\Rightarrow \kappa = 0$

$\Rightarrow f: N \to X$, $f(s) = (p(s), e_1(s), e_2(s), 0)$ and the curve $p: N \to S$ is a geodesic.