9-10. Elastic curves in space forms (2 dim'l).

$S, \langle , \rangle$ a Riemannian surface.

$\mathcal{F}(S) \xrightarrow{\pi} S$ the $O(2)$-bundle of o.n. frames on S.

ω^1, ω^2 the canonical 1-forms on $\mathcal{F}(S)$, $\omega_1^2 = -\omega_2^1$ the Levi-Civita conn. form.

$\omega^1, \omega^2, \omega_1^2$ is a coframe field on $\mathcal{F}(S)$.

$d\omega^1 = -\omega_2^1 \wedge \omega^2$
$d\omega^2 = -\omega_1^2 \wedge \omega^1$
$d\omega_1^2 = -R\,\omega^1 \wedge \omega^2$, $R = K \circ F$, K = Gauss curv.

$X = \mathcal{F}(S) \times \mathbb{R}$, κ the coordinate on $\mathbb{R}$.

$\omega^1, \omega^2, \omega_1^2, d\kappa$ is a coframe field on X.

$\mathcal{I}$ is the Pfaffian system generated by $W^* = \operatorname{span}\{\theta^1, \theta^2\}$,

where
$$\theta^1 = \omega^2$$
$$\theta^2 = \omega_1^2 - \kappa\omega$$
and the independence condition is $\omega = \omega^1 \neq 0$.

Structure eqns:
$$d\theta^1 = -\theta^2 \wedge \omega$$
$$d\theta^2 = (R + \kappa^2)\theta^1 \wedge \omega - d\kappa \wedge \omega + \kappa\,\theta^1 \wedge \theta^2$$
$$d\omega = \kappa\omega \wedge \theta^1 - \theta^1 \wedge \theta^2.$$

Consider a fcn $L(\kappa)$ on $\mathbb{R}$ and let
$$\varphi = L(\kappa)\omega$$
for the variational problem $\Phi(N, f) = \int_N f^*\varphi$.

Case $L(\kappa) = 1$ is the arc-length problem considered last time.

Case $L(\kappa) = \frac{1}{2}\kappa^2$ is the elastic curve problem.

To find the Euler-Lagrange equations at a solution (N, f) of $(\mathcal{I}, \omega)$, we first calculate
$$d(\varphi + \lambda_1\theta^1 + \lambda_2\theta^2) \mod \theta^1 \wedge \theta^2.$$
λ_1, λ_2 fcns on X to be determined.

$$d(\varphi+\lambda_1\theta^1+\lambda_2\theta^2) \equiv (L'(\kappa)-\lambda_2)\,d\kappa\wedge\omega + (d\lambda_1+(\kappa L(\kappa)-\lambda_2(R+\kappa^2))\omega)\wedge\theta^1$$
$$+(d\lambda_2+\lambda_1\omega)\wedge\theta^2 \quad \text{mod } \theta^1\wedge\theta^2 .$$

Exercise: Verify this calculation & show that it agrees with the calculation last time when $L=1$.

Take interior product with frame: $\frac{\partial}{\partial\kappa}$, $\frac{\partial}{\partial\theta^1}$, $\frac{\partial}{\partial\theta^2}$ (may skip $\frac{\partial}{\partial\omega}$)

(i.e. take u = each of these).

1) $\frac{\partial}{\partial\kappa}\lrcorner\, d(\varphi+\lambda_1\theta^1+\lambda_2\theta^2) \equiv (L'(\kappa)-\lambda_2)\omega = 0 \pmod N$

2) $f^*\left(\frac{\partial}{\partial\theta^1}\lrcorner\, d(\varphi+\lambda_1\theta^1+\lambda_2\theta^2)\right) \equiv -(d\lambda_1+(\kappa L-\lambda_2(R+\kappa^2))\omega) = 0 \ (N)$

3) $f^*\left(\frac{\partial}{\partial\theta^2}\lrcorner\, d(\varphi+\lambda_1\theta^1+\lambda_2\theta^2)\right) \equiv -(d\lambda_2+\lambda_1\omega) = 0 \pmod N$

Euler-Lagrange equations of $(\mathcal{I},\omega,\varphi)$ at $(N,f)\in\mathcal{V}(\mathcal{I},\omega)$.

~~Since~~ $f^*\omega$ is never 0 on N,

1) $\Rightarrow$ $\lambda_2 = L'(\kappa)$

$\therefore$ 3) $\Rightarrow$ $dL'(\kappa)+\lambda_1\omega=0$ & $\omega=ds$ $\Rightarrow$ $\lambda_1 = -L''(\kappa)\frac{d\kappa}{ds}$

Then 2) $\Rightarrow$ $\frac{d\lambda_1}{ds} = -\kappa L + L'(R+\kappa^2)$

$$\frac{d\lambda_1}{ds} = \frac{d}{ds}\left(-L''(\kappa)\frac{d\kappa}{ds}\right) = -L'''(\kappa)\left(\frac{d\kappa}{ds}\right)^2 - L''(\kappa)\frac{d^2\kappa}{ds^2}$$

$\therefore$ the Euler-Lagrange equation at (N,f) is

$$L'''(\kappa)\left(\frac{d\kappa}{ds}\right)^2 + L''(\kappa)\frac{d^2\kappa}{ds^2} + L'(\kappa)(R+\kappa^2) - \kappa L(\kappa) = 0,$$

an ODE on the curvature $\kappa(s)$ of the curve $p(s)$, where $f(s)=(p(s),e_1(s),e_2(s),\kappa(s))$.

For the elastic curve problem, $L(\kappa) = \frac{1}{2}\kappa^2$, so

$L'(\kappa) = \kappa$, $L''(\kappa) = 1$, and $L'''(\kappa) = 0$, so the E-L eqn is

$$\frac{d^2\kappa}{ds^2} + \kappa(R+\kappa^2) - \frac{\kappa^3}{2} = 0$$

is (E-L) $$\frac{d^2\kappa}{ds^2} + \frac{\kappa^3}{2} + \kappa R = 0$$

<u>Case R = constant.</u> Now $S, \langle , \rangle$ has constant ~~Gaussian~~ curvature.

If $\pi_1(S) = 0$ & $S, \langle , \rangle$ complete, it is called a <u>space form</u>.

($R=0$ Euclidean geometry, $R>0$ elliptic geom, $R<0$ hyperbolic geom).

If the curve $p(s)$ in S coming from the integral curve (N, f) (& R is constant) then has geodesic curvature satisfying (E-L),

$$\frac{d}{ds}\left(\left(\frac{d\kappa}{ds}\right)^2 + \frac{\kappa^4}{4} + \kappa^2 R \right) = 2\frac{d\kappa}{ds}\frac{d^2\kappa}{ds^2} + \kappa^3\frac{d\kappa}{ds} + 2\kappa R\frac{d\kappa}{ds}$$

$$= 2\frac{d\kappa}{ds}\left(\frac{d^2\kappa}{ds^2} + \frac{\kappa^3}{2} + \kappa R\right) = 0 \text{ on } N.$$

$$\Rightarrow \quad (*) \quad \left(\frac{d\kappa}{ds}\right)^2 + \frac{\kappa^4}{4} + \kappa^2 R = c = \text{constant on } N.$$

In a space form, for any function κ on N $\exists$ unique curve $p: N \to S$ up to congruence, whose geodesic curvature is κ.

Thus, for any solution $\kappa(s)$ of $(*)$, the curve $p: N \to S$ with geodesic curv κ is an elastic curve.

<u>Exercise</u> A constant fcn κ satisfies (E-L) iff $\kappa = 0$ or $\kappa = \pm\sqrt{-2R}$, i.e. $p(s)$ is a unit speed geodesic or, where $R<0$, say $R=-1$, $\kappa = \pm\sqrt{2}$.

What are the curves of $\kappa = \sqrt{2}$ in the hyperbolic plane.

To solve (*), solve for $\frac{d\kappa}{ds}$ & then separate variables:

$$\frac{d\kappa}{\sqrt{c-\kappa^2 R-\frac{\kappa^4}{4}}} = ds.$$

define $\kappa(s)$ by

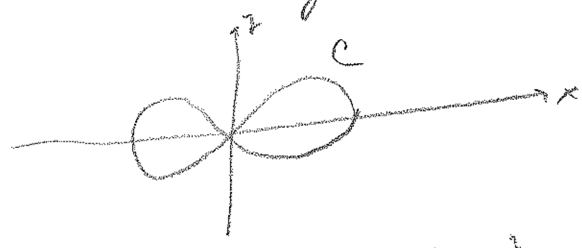
$$\int_{\kappa_0}^{\kappa(s)} \frac{dt}{\sqrt{c-t^2 R-\frac{t^4}{4}}} = s + c_1$$

Then $\kappa(s)$ is the restriction to $\mathbb{R}\subset\mathbb{C}$ of a doubly periodic meromorphic function on all of $\mathbb{C}$ (elliptic function).

<u>algebraic-geometric</u> approach.

Let $C = \{(x,y)\in\mathbb{C}^2 : y^2 + \frac{x^4}{4} + x^2R - c = 0\}$.

<u>Case $c=0$</u>. $C = \{(x,y)\in\mathbb{C}^2 : y^2+\frac{x^4}{4}+x^2R=0\}$ is a smooth curve with ordinary double point at $(0,0)$



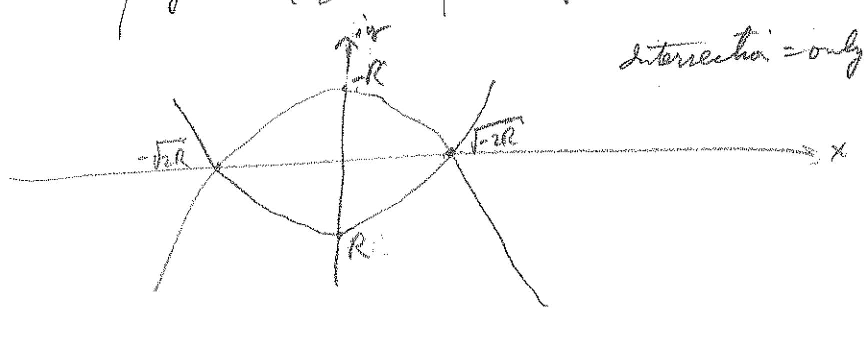
<u>Case $c=-R^2$</u>. $C = \{(x,y)\in\mathbb{C}^2 : y^2+\left(\frac{x^2}{2}+R\right)^2 = 0\}$

$$\left(y+i\left(\tfrac{x^2}{2}+R\right)\right)\left(y-i\left(\tfrac{x^2}{2}+R\right)\right)$$

$= \{y = i\left(\frac{x^2}{2}+R\right)\} \cup \{y = -i\left(\frac{x^2}{2}+R\right)\}$ imaginary parabolas

intersection = only two real points.

<u>R<0 picture</u>



all other cases i.e. $c \notin \{0, -R^2\}$, then C is an elliptic curve parametrized by an elliptic function $\kappa(s)$ as $s \mapsto (\kappa(s), \kappa'(s))$

where modulus depends on R & c.

In the exceptional cases, each component of C is rational, given parametrically by $s \mapsto (\kappa(s), \kappa'(s))$, where $\kappa'(s)$ is rational fcn of s. In these cases the E-L eqns are solved by elementary functions.

Closed elastic curves in Euclidean plane.

An *elastic curve* in a Riemannian surface $S, \langle , \rangle$ of Gaussian curvature R is an immersed curve $p(s)$, parametrized by arc-length (so $|\dot{p}| = 1$) whose *geodesic curvature* $\kappa(s)$ satisfies

$$(E\text{-}L) \qquad \frac{d^2\kappa}{ds^2} + \left(\frac{\kappa^3}{2} + \kappa R\right) = 0$$

Case $R = 0$, the Euclidean plane $S = \mathbb{R}^2$, $\langle , \rangle =$ dot product.

Now $(E\text{-}L)$ is $\quad \dfrac{d^2\kappa}{ds^2} + \dfrac{\kappa^3}{2} = 0$

Phase plane: Let $x = \kappa$, $y = \dot{\kappa}$ and the ODE system

$$(1) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -\dfrac{x^3}{2} \end{cases}.$$

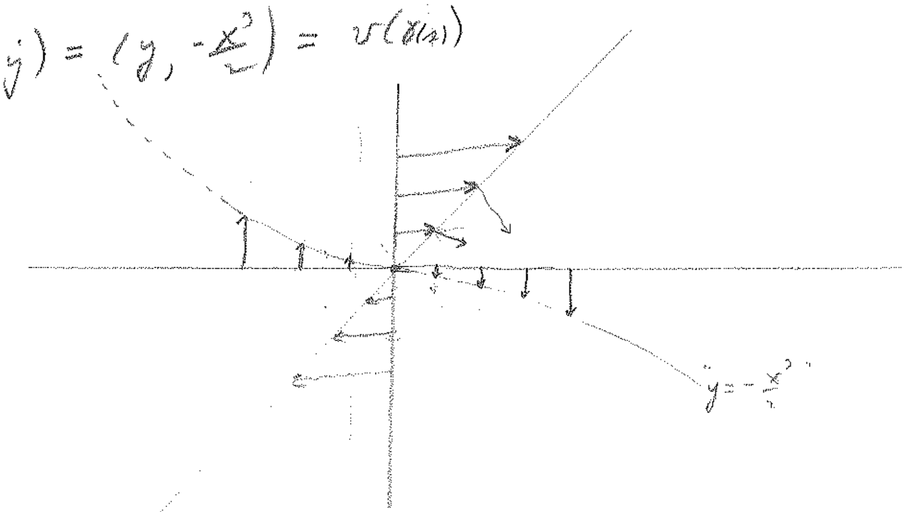
given by the vector field $v = y \dfrac{\partial}{\partial x} - \dfrac{x^3}{2} \dfrac{\partial}{\partial y}$ on the xy-plane.

I.e., a solution curve $\gamma(s) = (x(s), y(s))$ of (1) is integral curve of v,

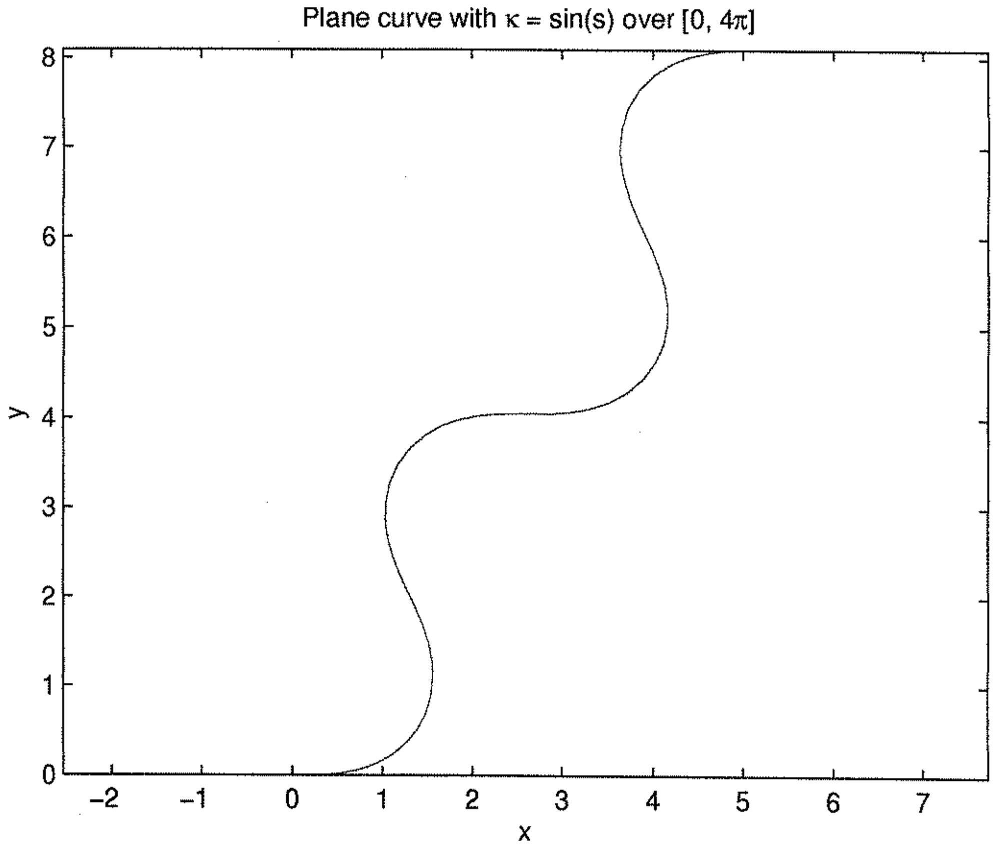
$$\dot{\gamma}(s) = (\dot{x}, \dot{y}) = \left(y, -\frac{x^3}{2}\right) = v(\gamma(s))$$

Picture of v:

$v(x,y) = 0$ iff $x = y = 0$.
Only singular point of v.



Here is a curve in the Euclidean plane whose curvature is $\kappa = \sin s$, plotted over the range $[0, 4\pi]$.



$p(s)$ parametrized by arclength s.

$p(0) = (0,0)$, $\dot{p}(0) = \varepsilon_1 = (1,0)$.

$\dot{p} = e_1$, $\ddot{p} = \dot{e}_1 = \kappa(s)\, e_2$, where $\kappa(s) = \sin s$

Produced with Curve1.m in Matlab with function file F.m

in Talks/TorinoLectures2010

The critical point $(0,0)$ of v corresponds to elastic curves with $\kappa = 0$, that is, lines in $\mathbb{R}^2$.

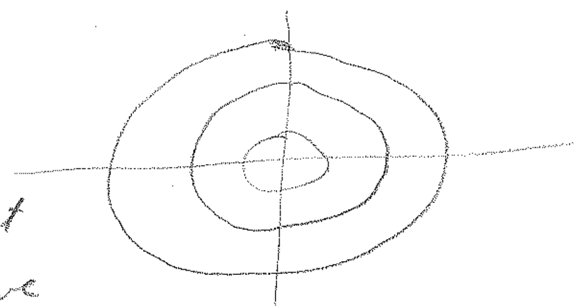
Now $\quad \ddot{\kappa} + \frac{\kappa^3}{2} = 0 \qquad \left(\dot{\kappa} = \frac{d\kappa}{ds}\right)$

$$\Rightarrow \left(\frac{(\dot{\kappa})^2}{2} + \frac{\kappa^4}{8}\right)^{\cdot} = \dot{\kappa}\left(\ddot{\kappa} + \frac{\kappa^3}{2}\right) = 0$$

$$\Rightarrow (\dot{\kappa})^2 + \frac{\kappa^4}{4} = c^2 \geq 0 \text{ is constant.}$$

Thus, the integral curves of v must lie on level curves of

$$y^2 + \frac{x^4}{4} = c^2$$



Since $(0,0)$ is the only singular point of v, every other integral curve must be periodic, going around one of these level curves.

$\therefore$ $\kappa(s)$ must be a periodic function of s, say

(a) $\kappa(s+\ell) = \kappa(s)$, $\forall s$, some $\ell > 0$.

But periodic curvature does not imply periodic curve.

See attached graph where $\kappa(s) = \sin s$.

Write $p(s) = (x(s), y(s))$

Frenet frame $\quad \dot{p}(s) = e_1(s) = (\cos\theta(s), \sin\theta(s))$

$$\ddot{p} = \dot{e}_1 = \kappa(s)\, e_2(s) = \underbrace{(-\sin\theta, \cos\theta)}_{= e(s)}\dot{\theta}(s) \Rightarrow \kappa(s) = \dot{\theta}(s)$$

If $p(s)$ were a closed curve, ~~then~~ also

$$p(s+\ell) - p(s) = 0 \quad \forall s.$$

$\therefore \quad 0 = \underset{\underset{e_1(s+\ell)}{\shortparallel}}{\dot{\rho}(s+\ell)} - \underset{\underset{e_1(s)}{\shortparallel}}{\dot{\rho}(s)} \Rightarrow \cos\theta(s+\ell) - \cos\theta(s) = 0 = \sin\theta(s+\ell) - \sin\theta(s)$

$\Rightarrow$ (b) $\theta(s+\ell) - \theta(s) = 2\pi w$, for some $w \in \mathbb{Z}$.

Then $\int_0^\ell \kappa(s)\,ds = \theta(\ell) - \theta(0) = 2\pi w \Rightarrow w$ is the rotation index of ρ.

And $\rho(s+\ell) - \rho(s) = \int_s^{s+\ell} \underbrace{\dot{\rho}(t)}_{\underset{e_1(t)}{\shortparallel}}\,dt$

$\Rightarrow \quad \frac{d}{ds}\left(\rho(s+\ell) - \rho(s)\right) = e_1(s+\ell) - e_1(s)$