# 11. The Euler-Lagrange EDS.

Given a variational problem $(\mathcal{I}, \omega, \varphi)$ on $X$,

write the Euler-Lagrange equations at $(N, f) \in V(\mathcal{I}, \omega)$

$$f^*(u \lrcorner\, d(\varphi + \Theta)) = 0 \quad , \forall\, u \in C^\infty(N, TX) \quad , \text{ some } \Theta \in C^\infty(N, W^*)$$

as an EDS $(\mathcal{J}, \omega)$ on an associated man. $Y$.

In our application of the following construction, the closed 2-form $\Psi$ will be $d(\varphi + \Theta)$.

Let $\Psi$ be a closed 2-form on a manifold $Z$.

The rank of $\Psi$ at a point $x \in Z$ is the integer $\rho(x)$ s.t.

$$\Psi(x)^{\rho(x)} = \overbrace{\Psi(x) \wedge \dots \wedge \Psi(x)}^{\rho(x)\text{ terms}} \neq 0$$

$$\& \quad \Psi(x)^{\rho(x)+1} = 0$$

To put the discussion in context, if the rank $\rho$ of $\Psi$ is constant on $Z$, then Darboux's Theorem says each point of $Z$ has a coordinate chart $(U, z)$ about such that

$$z = (u_1, \dots, u_\rho, v^1, \dots, v^\rho, z^{2\rho+1}, \dots, z^n) \qquad (n = \dim Z)$$

such that

$$\Psi = du_1 \wedge dv^1 + \dots + du_\rho \wedge dv^\rho \qquad \text{on } U.$$

In our applications, the rank of $\Psi$ will generally not be constant.

<u>Def.</u> The <u>Cartan system</u> $C(\Psi)$ is the Pfaffian system on $Z$ generated by all 1-forms

$$\{ u \lrcorner \Psi : u \in C^\infty(Z, TZ) \}.$$

Let $\omega$ be a 1-form on $Z$ & consider the EDS $(C(\Psi), \omega)$ on $Z$.

For any nonzero vector $v \in T_pZ$, let $[v] \subset T_pZ$ denote the 1-dim'l subspace spanned by $v$.

Recall that an <u>integral element</u> of $(C(\Psi), \omega)$ is a pair $(p, [v])$, where $0 \neq v \in T_pZ$, such that $(u \lrcorner \Psi_{(p)})(v) = 0 \ \forall u \in C^\infty(Z, TZ)$ & $v \lrcorner \omega_{(p)} \neq 0$.

<u>Exercise</u> $V$ a vector space over $\mathbb{R}$, $\Psi \in \Lambda_2 V^*$.

For any $u \in V$, $u \lrcorner \Psi \in V^*$, whose value on $v \in V$ is

$$(u \lrcorner \Psi)(v) = \Psi(u, v) = \langle \Psi, u \wedge v \rangle \quad \text{(pairing)} \quad \langle \Lambda_2 V^*, \Lambda_2 V \rangle \to \mathbb{R}.$$

We use the pairing given on decomposable elts by

$$\langle \theta^1 \wedge \dots \wedge \theta^k, v_1 \wedge \dots \wedge v_k \rangle = \det(\theta^i(v_j)).$$

Thus, $0 = (u \lrcorner \Psi_{(p)})(v) \quad \forall u \in T_pZ$

$$= \Psi_{(p)}(u \wedge v) = -\Psi_{(p)}(v \wedge u) = -(v \lrcorner \Psi_{(p)})(u), \quad \forall u \in T_pZ$$

$\Leftrightarrow \quad v \lrcorner \Psi_{(p)} = 0.$

Hence, $(p, [v])$ is an integral element of $(C(\Psi), \omega)$ iff

$$v \lrcorner \Psi_{(p)} = 0 \quad \& \quad v \lrcorner \omega_{(p)} \neq 0.$$

Def. $V(C(\Psi), \omega)$ is the set of all integral elements of $(C(\Psi), \omega)$.

Projection map $\pi : V(C(\Psi), \omega) \to Z$, $\pi(p, [v]) = p$.

In general, $\pi$ is not surjective. That is, $\exists$ points $p \in Z$ at which there are no integral elements. $(p, [v])$.

Let $Z_1$ = the image of $\pi$. Assume it is a smooth subman of $Z$.
= set of all points of $Z$ at which $(C(\Psi), \omega)$ has an int. elt.

Let $C_1(\Psi) = C(\Psi)_{Z_1} = i^* C(\Psi)$ where $i : Z_1 \hookrightarrow Z$ inclusion.

$$\omega = \omega_{Z_1} \in A^1(Z_1)$$

$$\Psi_1 = \Psi_{Z_1} \in A^2(Z_1)$$

Thus $(C_1(\Psi), \omega)$ is the restriction to $Z_1$ of $(C(\Psi), \omega)$.

Note $C(\Psi_1) \subset C_1(\Psi)$, but equality may not hold.

Fact. $(N, f)$ is an integral manifold of $(C(\Psi), \omega)$ iff it is an integral manifold of $(C_1(\Psi), \omega)$.

Proof. "$\Leftarrow$" clear. "$\Rightarrow$" If $f : N \to Z$ is an integral manifold of $(C(\Psi), \omega)$, then $\forall\, x \in N$, $(f(x), [f_* \frac{\partial}{\partial x}])$ is an integral elt, so $p \in Z_1$ $\therefore$ $f(N) \subset Z_1$, and $\forall$ $\varphi_1 \in C_1(\Psi)$, $\varphi_1 = \varphi_{Z_1}$, some $\varphi \in C(\Psi)$ & $f^* \varphi_1 = f^* \varphi = 0$.

Def. $\pi : V(C_1(\Psi), \omega) \to Z_1$ might not be surjective.

Let $Z_2$ = image of $\pi$ (assume subman). $Z_2 \subset Z_1 \subset Z$.

$C_2(\Psi) = C(\Psi)_{Z_2}$, $\omega = \omega_{Z_2}$

$\Psi_2 = (\Psi_1)_{Z_2} = \Psi_{Z_2}$.

At some step the process stops when $Z_{k_0+1} = Z_{k_0}$, that is, the projection $\pi(C_{k_0}(\Psi), \omega) \to Z_{k_0}$ is surjective for the first time.

Let $Y = Z_{k_0}$

$$J = C_{k_0}(\Psi) = C(\Psi)_{Y}. \quad (= i_{k_0}^* C(\Psi), \ i_{k_0}: Y \hookrightarrow Z)$$

$$\omega = \omega_{Y}$$

Note: 1) $C(\Psi_Y) \subset J$, but equality may not hold.

2). $\pi: V(J,\omega) \to Y$ is surjective.

3). The integral manifolds of $(C(\Psi), \omega)$ in $Z$ coincide with those of $(J, \omega)$ in $Y$.

Def i) $(J,\omega)$ on $Y$ called the Euler-Lagrange EDS associated to $(I, \omega, \varphi)$ on $X$.
ii) Call $Y$ the momentum space associated to $(I, \omega, \varphi)$.

Theorem Let $(I, \omega, \varphi)$ be a variational problem on $X$, with Euler-Lagrange equations at $(N, f) \in V(I, \omega)$

$$f^*(u \lrcorner d(\varphi + \theta)) = 0, \quad \forall u \in C^\infty(N, TX) \quad (*)$$

for some section $\theta$ of $W^*$ ($I$ = Pfaffian system gen. by sections of $W^* \subset T^*X$).

Let $Z = W^*$, $\psi = \varphi + \theta$, $\Psi = d(\varphi + \theta)$ on $Z$, and let $(J, \omega)$ be the EDS on $Y$ constructed as above for the closed 2-form $\Psi$ on $Z$.

There is a natural 1:1 correspondence between the $(N, f) \in V(I, \omega)$ satisfying the E-L equations $(*)$ and the integral manifolds of $(J, \omega)$ in $Y$.

Before proving this Theorem, we'll construct $Y$ and $(J, \omega)$ for the classical var. prob.