12. Euler-Lagrange EDS for the classical var. problem.

$X = J^1(\mathbb{R}, M)$, local coordinates $x, y^\alpha, \dot y^\alpha$.

I = Pfaffian system generated by sections of $W^* \subset T^*X$,

where in local coordinates $\{\theta^\alpha = dy^\alpha - \dot y^\alpha dx\}$ is a framing of W^*.

i.e. $W^*_{(p)} = \{\theta^\alpha_{(p)}\}$, $\forall\, p \in$ domain of $(x, y, \dot y)$.

$\omega = dx$.

$\varphi = \mathcal{L}(x, y, \dot y)\,\omega$

$Z = W^* \subset T^*X = X \times \mathbb{R}^m$ locally, $m = \dim M$.

Local coords (x, y^α) on $\mathbb{R} \times M$ induce local coordinates

$(x, y^\alpha, \dot y^\alpha, \lambda_\alpha)$ on Z, where for any point $(p, \theta) \in Z$,

$\lambda_\alpha(p, \theta) \in \mathbb{R}$ is determined by $\theta \in T^*_pX$, $\theta = \lambda_\alpha \theta^\alpha_{(p)}$.

$\psi = \varphi + \theta = L(x, y, \dot y)\,\omega + \lambda_\alpha \theta^\alpha$, 1-form on Z

$$\boxed{\Psi = d\psi = (L_{\dot y^\alpha} - \lambda_\alpha)\, d\dot y^\alpha \wedge \omega + (d\lambda_\alpha - L_{y^\alpha}\omega) \wedge \theta^\alpha}$$, closed 2-form on Z.

Exercise Find the rank $\rho(z)$ of $\Psi_{(z)}$, $\forall\, z \in Z$.

Show $\rho(z) = m+1$ if $z \notin Z_1$, $\rho(z) = m$ if $z \in Z_1$

[Soln: let $\sigma = (L_{\dot y^\alpha} - \lambda_\alpha) d\dot y^\alpha$; $\mu_\alpha = d\lambda_\alpha - L_{y^\alpha}\omega \in A^1(Z)$

Then $\mu_1 \wedge \dots \wedge \mu_m \neq 0$ at every point of Z & $\sigma_{(z)} = 0$ iff $z \in Z_1$.

$\Psi^m = c_1\, \sigma \wedge \omega \wedge (\mu_1 \wedge \theta^1 \wedge \dots \wedge \widehat{\mu_\alpha \wedge \theta^\alpha} \wedge \dots \wedge \mu_m \wedge \theta^m) + c_2\, \mu_1 \wedge \theta^1 \wedge \dots \wedge \mu_m \wedge \theta^m \neq 0$

$\Psi^{m+1} = c_3\, \sigma \wedge \omega \wedge \mu_1 \wedge \theta^1 \wedge \dots \wedge \mu_m \wedge \theta^m$, $c_1, c_2, c_3 \neq 0$ constants.]

Coframe on Z is $dx, \theta^\alpha, d\dot y^\alpha, d\lambda_\alpha$

Dual frame $\frac{\partial}{\partial x}, \frac{\partial}{\partial \theta^\alpha}, \frac{\partial}{\partial \dot y^\alpha}, \frac{\partial}{\partial \lambda_\alpha}$

$C(\Psi)$ is generated by the Pfaffian eqns:

i) $\dfrac{\partial}{\partial \lambda_\alpha} \lrcorner \Psi = \theta^\alpha = 0$ (this is $\mathcal{I}$)

ii) $\dfrac{\partial}{\partial \dot{y}^\alpha} \lrcorner \Psi = (L_{\dot{y}^\alpha} - \lambda_\alpha)\, dx = 0$

iii) $\dfrac{\partial}{\partial \theta^\alpha} \lrcorner \Psi = L_{y^\alpha} dx - d\lambda_\alpha = 0$

iv) $\dfrac{\partial}{\partial x} \lrcorner \Psi = (\lambda_\alpha - L_{\dot{y}^\alpha}) d\dot{y}^\alpha - L_{y^\alpha} \theta^\alpha = 0$

$(C(\Psi), \omega)$ is the Pfaffian system on Z generated by

(*) $\left\{ \theta^\alpha,\ (L_{\dot{y}^\alpha} - \lambda_\alpha) dx,\ d\lambda_\alpha - L_{y^\alpha} dx,\ (\lambda_\alpha - L_{\dot{y}^\alpha}) d\dot{y}^\alpha \right\}$

For a point $z \in Z$, a basis of $T_z Z$ is $\frac{\partial}{\partial x}, \frac{\partial}{\partial \theta^\alpha}, \frac{\partial}{\partial \dot{y}^\alpha}, \frac{\partial}{\partial \lambda_\alpha}$ so

$v = a\frac{\partial}{\partial x} + b^\alpha \frac{\partial}{\partial \theta^\alpha} + c^\alpha \frac{\partial}{\partial \dot{y}^\alpha} + e^\alpha \frac{\partial}{\partial \lambda_\alpha}$, coefs in $\mathbb{R}$ is an arbitrary elt of $T_z Z$,

$(p, [v])$ is an <u>integral element</u> of $(C(\Psi), \omega)$

$\Rightarrow$ $a = v \lrcorner \omega \neq 0$ and $(L_{\dot{y}^\alpha} - \lambda_\alpha)\, \omega(v) = 0 \Rightarrow$ no int elt if $\lambda_\alpha - L_{\dot{y}^\alpha} \neq 0$

$\therefore$ $Z_1 \subset \{ z \in Z : \lambda_\alpha - L_{\dot{y}^\alpha} = 0 \}$.

If $z \in Z_1$, then $C(\Psi)_{z_1} = \{ \theta^\alpha_{(z)}, (d\lambda_\alpha - L_{y^\alpha}\omega)_{(z)} \}$, so

$[z, [v]]$ is an integral elt iff $b^\alpha = 0$ & $e^\alpha = L_{y^\alpha}(z) a$, $\alpha = 1, \dots, m$.

Hence the image of $\pi : V(C(\Psi), \omega) \to Z$ is $Z_1 = \{ \lambda_\alpha = L_{\dot{y}^\alpha} : \alpha = 1, \dots, m \}$

On Z_1, $d\lambda_\alpha = L_{\dot{y}^\alpha \dot{y}^\beta} d\dot{y}^\beta \mod \{ \theta^\alpha, dx \}$ (i.e. $i^*(d\lambda_\alpha = L_{\dot{y}^\alpha \dot{y}^\beta} d\dot{y}^\beta) \mod i^*\theta^\alpha, i^*dx$

$i : Z_1 \hookrightarrow Z$.

Now $\omega, \theta^\alpha, d\lambda_\alpha$ will be a coframe field on Z_1 iff

$0 \neq \omega \wedge \theta^1 \wedge \dots \wedge \theta^m \wedge d\lambda_1 \wedge \dots \wedge d\lambda_m = \det(L_{\dot{y}^\alpha \dot{y}^\beta})\, \omega \wedge \theta^1 \wedge \dots \wedge \theta^m \wedge d\dot{y}^1 \wedge \dots \wedge d\dot{y}^m$

i.e. iff $\det(L_{\dot{y}^\alpha \dot{y}^\beta}) \neq 0$ at each point of Z_1.

Def. The functional $\int_N L(x, y^\alpha, \dot y^\alpha)\,dx$ is <u>non-degenerate</u> if $\det(L_{\dot y^\alpha \dot y^\beta}) \neq 0$ (on X, or some unspecified subset of X).

As we'll see, this is the assumption that the variational problem (I, ω, φ) is non-degenerate (To be defined later) (even strongly non-degenerate).

Then $\{dx, \theta^\alpha, d\lambda_\alpha\}$ is a coframe field on Z_1, i.e. on Z_1 can solve for the $d\dot y^\alpha$'s in terms of the $d\lambda_\alpha$

$$\psi_{Z_1} \quad (= i^*\psi) = \mathcal{L}\,dx + \lambda_\alpha \theta^\alpha$$

$$\Psi_{Z_1} = (d\lambda_\alpha - L_{y^\alpha}\omega)\wedge\theta^\alpha$$

$\therefore$ $C(\Psi_{Z_1})$ is generated by $\{\theta^\alpha, d\lambda_\alpha - L_{y^\alpha}\omega\}$ on Z_1

$C_1(\Psi) \overset{df}{=} C(\Psi)_{Z_1}$ is generated by $\{\theta^\alpha, d\lambda_\alpha - L_{y^\alpha}dx\}$ as well (by ⊛)

$= C(\Psi_{Z_1})$

$\psi_{Z_1}\wedge(\Psi_{Z_1})^m = m!\,\mathcal{L}\,dx\wedge d\lambda_1\wedge\theta^1\wedge\dots\wedge d\lambda_m\wedge\theta^m \neq 0$ at every point of Z_1

Moreover, $Y = Z_1$. (Obvious, once one grasps the definition of int elt).

Proof.
We must show that at any point $z_1 \in Z_1$, there is an integral element $(z_1, [v])$ of $(C_1(\Psi), \omega)$.
Any $v = \frac{\partial}{\partial x} + L_{y^\alpha}(z_1)\frac{\partial}{\partial \lambda_\alpha}$ defines such an integral element.

$\therefore$ $Y = Z_1$ and $J = C_1(\Psi)$ is generated by $\{\theta^\alpha, d\lambda_\alpha - L_{y^\alpha}dx\}$.

Remark $\psi_Y = \psi_{Z_1} = \mathcal{L}dx + \lambda_\alpha \theta^\alpha = (\mathcal{L} - \lambda_\alpha \dot{y}^\alpha)dx + \lambda_\alpha dy^\alpha$

$= -H dx + \lambda_\alpha dy^\alpha$ is in the Pfaff-Darboux form in the coords $x, y^\alpha, \lambda_\alpha$ on Z_1.

where $H = \lambda_\alpha \dot{y}^\alpha - \mathcal{L}(x, y, \dot{y})$ is a function on Y, called the Hamiltonian.

Since $\lambda_\alpha = \mathcal{L}_{\dot{y}^\alpha}$ on Y,

$H = \dot{y}^\alpha \mathcal{L}_{\dot{y}^\alpha} - \mathcal{L}$ is the classical Hamiltonian.

For this classical variational problem, H is actually a function on X, but in general H is a function on $Y = X \times W^*$ that does not descend to X.

For the special case of the mechanical system

$$\mathcal{L} = T(y, \dot{y}) - U(y), \qquad T(y, \dot{y}) = \tfrac{1}{2} g_{\alpha\beta}(y)\, \dot{y}^\alpha \dot{y}^\beta$$

we have $\mathcal{L}_{\dot{y}^\alpha} = g_{\alpha\beta} \dot{y}^\beta$, so

$$H = \dot{y}^\alpha g_{\alpha\beta} \dot{y}^\beta - \tfrac{1}{2} g_{\alpha\beta} \dot{y}^\alpha \dot{y}^\beta + U = T(y, \dot{y}) + U(y) = \text{"total energy"}.$$