13. Good form, derived flag, and Cauchy characteristics.

pp 44–52.

Let $(\mathcal{I}, \omega)$ be a Pfaffian EDS on X generated by sections of a subbundle $W^* \subset T^*X$.

Locally, if $\theta^1, \ldots, \theta^s$ is a frame field of W^* (basis of $W^*_{(x)}$ $\forall x \in U \subseteq X$) then $\mathcal{I}$ is generated by $\{\theta^\alpha\}$ = the algebraic ideal generated by the θ^α. $\subseteq A^*(X)$

Def $(\mathcal{I}, \omega)$ is a Pfaffian system in good form if $\exists$ 1-forms π^α s.t. (1) $d\theta^\alpha \equiv -\pi^\alpha \wedge \omega \mod \{\theta^\alpha\}$.

Remark Complete to a local coframe: $\omega, \theta^\alpha, \eta^i$ of X.

Then $d\theta^\alpha \equiv -\pi^\alpha \wedge \omega + A_{ij}\,\eta^i \wedge \eta^j \mod \{\theta^\alpha\}$ in general.

Good form means the term $A_{ij}\,\eta^i \wedge \eta^j$ does not occur.

Example The canonical Pfaffian system on $X = J^1(\mathbb{R}, M)$ given in local coords x, y^α on M by $\theta^\alpha = dy^\alpha - \dot{y}^\alpha dx = 0$, $\omega = dx \neq 0$ is in good form, $\because$ $d\theta^\alpha = -d\dot{y}^\alpha \wedge dx$, so $\pi^\alpha = d\dot{y}^\alpha$.

Def. The first derived system of W^* is the subbundle $W_1^* \subset W^*$ defined by $\{\theta \in C^\infty(X, W^*) : d\theta \equiv 0 \mod C^\infty(X, W^*)\}$

It is a subbundle if its rank is constant. In fact:

Let $W^* \wedge T^*X$ be the image of $W^* \otimes T^*X$ under wedge product

Let $\delta : W^* \to \Lambda^2 T^*X / W^* \wedge T^*X$ be induced by exterior differentiation d. Then δ is $C^\infty(X)$-linear, which means it is a bundle map.

$$(W_1^*)_{(p)} = \ker \delta_{(p)} \subset W^*_{(p)} \quad , \text{ fiber over } p \in X.$$

Γ δ $C^\infty(X)$-linear: $d(f\theta) = df\wedge\theta + f d\theta \equiv f d\theta \mod W^*\wedge T^*X$

$\therefore$ for any θ, $(\delta\theta)_{(p)}$ depends only on $\theta_{(p)}$.

In fact, $\theta^1, \ldots, \theta^s$ a frame field of W^*, $\theta = \lambda_\alpha\theta^\alpha$, $\theta_{(p)} = \lambda_\alpha(p)\theta^\alpha_{(p)}$

Then $\delta\theta = \lambda_\alpha\delta\theta^\alpha \Rightarrow (\delta\theta)_{(p)} = \lambda_\alpha(p)(\delta\theta^\alpha)_{(p)}$ depends only on $\lambda_\alpha(p)$. $\rfloor$

δ is the derived mapping associated to the Pfaffian system W^*.

$\operatorname{Ker}\delta = W_1^* \subset W^*$ is the 1st <u>derived system</u> of W^*.

If $L^* = \operatorname{span}\{\theta^1, \ldots, \theta^s, \omega\}$, then $(\mathcal{I}, \omega)$ is in good form means

$$\delta(W^*) \subset L^*\wedge T^*X / W^*\wedge T^*X$$

<u>Exercise</u> If a Pfaffian system $(\mathcal{I}, \omega)$ on X is in good form, and if $i: Y \hookrightarrow X$ is a submanifold such that $\omega_Y = i^*\omega \neq 0$, then the restriction of $(\mathcal{I}, \omega)$ to Y $((i^*\mathcal{I}, i^*\omega))$ is in good form.

<u>Remark</u>. Continue to get <u>derived flag</u> $W^* \supset W_1^* \supset \ldots \supset W_k^* \supset \ldots$. Illustrated in next example.

Def. If a Pfaffian system $(\mathcal{I}, \omega)$ on X is in good form with first derived system $W_1^* \subset W^* \subset T^*X$, then the <u>Cartan integer</u> $s_1 = s_1(\mathcal{I}, \omega)$ is the rank of W^*/W_1^*.

<u>Exercise</u> Prove that s_1 is the number of linearly indep't 1-forms $\pi^\alpha \in C^\infty(X, T^*X/L^*)$ appearing in (1).

Γ Soln. $\delta(W^*) = \operatorname{span}\{[\pi^\alpha\wedge\omega]\}$, $[\pi^\alpha\wedge\omega]$ = equiv. class in $\Lambda^2 T^*X / W^*\wedge T^*X$

$\therefore s_1 = \operatorname{rank}\delta(W^*) = \#$ linearly indep't elts $\{[\pi^\alpha\wedge\omega]\}$ in $\Lambda^2 T^*X / W^*\wedge T^*X$

$= \#$ linearly indep't elts $\{[\pi^\alpha]\}$ in T^*X/L^*,

$\because$ $\sum t_\alpha(\pi^\alpha\wedge\omega) \in W^*\wedge T^*X \iff (\sum t_\alpha\pi^\alpha)\wedge\omega \in L^*$.

(p. 54) Theorem I.c.16 A general $[v] \in T_N(V(\mathcal{I},\omega))$ is specified by s_1 arbitrary functions of one variable plus a certain number of constants.

Proof. (Bottom p48 thru p54) Sketch 1) Assume no Cauchy characteristics
2) Reduction by prolongation.

$(\mathcal{I},\omega)$ on X assumed a Pfaffian EDS in good form.

$\mathcal{I}$ generated by θ^α, $\alpha = 1,..,s$, $W^* = \text{span}\{\theta^\alpha\}$.

$W^* \supset W_1^*$ first derived system. Assume

$W_1^* = \text{span}\{\theta^1, .., \theta^{s-s_1}\}$, $\quad 1 \le \rho, \sigma \le s - s_1$

Let $s - s_1 \le \mu, \nu \le s$.

Then i) $d\theta^\rho \equiv 0 \mod \{\theta^\alpha\}$

ii) $d\theta^\mu \equiv -\pi^\mu \wedge \omega \mod \{\theta^\alpha\}$.

$\{\pi^\mu\}$ linearly indep't mod $L^* = \text{span}\{\omega, \theta^\alpha\}$.

Suppose $\{\omega, \theta^\alpha, \pi^\mu\}$ is a coframe field on X.

This means no Cauchy characteristics (see below, if time).

Dual frame $\frac{\partial}{\partial\omega}, \frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\pi^\mu}$ on X.

For an integral manifold (N, f) of $(\mathcal{I},\omega)$ we want to determine the "size" of the set of vector fields v along f $(v \in C^\infty(N, f^*TX))$ whose extension to X satisfies the variational equations $(i.e.\ f^*(\ ) = 0)$.

$$v \lrcorner d\theta^\alpha + d(v \lrcorner \theta^\alpha) \equiv 0 \mod N, \quad \alpha = 1,..,s$$

Let $W^* \wedge W^*$ be the $C^\infty(X)$-span of $\theta^\alpha \wedge \theta^\beta$

Let $L^* \wedge L^*$ be the $C^\infty(X)$-span of $\theta^\alpha \wedge \theta^\beta$, $\theta^\alpha \wedge \omega$.

Then i) & ii) say

iii) $d\Theta^\rho \equiv C^\rho_{\alpha\mu}\,\Theta^\alpha\wedge\pi^\mu \mod L^*\wedge L^*$, some fcns $C^\rho_{\alpha\mu}$.

iv) $d\Theta^\mu \equiv -\pi^\mu\wedge\omega + D^\mu_{\alpha\nu}\,\Theta^\alpha\wedge\pi^\nu \mod L^*\wedge L^*$, $D^\mu_{\alpha\nu}\in C^\infty(X)$

By an argument involving prolongation, which we must omit, we may assume the functions $C^\rho_{\alpha\mu}$ & $D^\mu_{\alpha\nu}$ are all zero, so i) & ii) may be assumed to be

$$\begin{cases} d\Theta^\rho \equiv 0 \mod L^*\wedge L^* \\ d\Theta^\mu \equiv -\pi^\mu\wedge\omega \mod L^*\wedge L^* \end{cases}$$

i.e.

$$(*)\ \begin{cases} d\Theta^\rho \equiv -E^\rho_\alpha\,\Theta^\alpha\wedge\omega \mod W^*\wedge W^* \\ d\Theta^\mu \equiv -\pi^\mu\wedge\omega + B^\mu_\alpha\,\Theta^\alpha\wedge\omega \mod W^*\wedge W^* \end{cases},\qquad E^\rho_\alpha,\ B^\mu_\alpha\in C^\infty(X).$$

Replacing π^μ by $\tilde\pi^\mu = \pi^\mu - B^\mu_\alpha\Theta^\alpha$, we have $\omega, \Theta^\alpha, \tilde\pi^\mu$ still a coframe field on X, and $(*)$ becomes

i)′ $d\Theta^\rho \equiv -E^\rho_\alpha\,\Theta^\alpha\wedge\omega \quad (W^*\wedge W^*)$

ii)′ $d\Theta^\mu \equiv -(\tilde\pi^\mu + B^\mu_\alpha\Theta^\alpha)\wedge\omega + B^\mu_\alpha\,\Theta^\alpha\wedge\omega = -\tilde\pi^\mu\wedge\omega \quad (W^*\wedge W^*)$

Write π^μ ∘

Given $(N,f)\in\mathcal{V}(I,\omega)$ and any smooth v.f. v along f, then its extension to X is of the form

$$v = A\frac{\partial}{\partial\omega} + B^\alpha\frac{\partial}{\partial\Theta^\alpha} + C^\mu\frac{\partial}{\partial\pi^\mu},\qquad A, B^\alpha, C^\mu\in C^\infty(X).$$

Since $(A\frac{\partial}{\partial\omega})\circ f$ is tangent to N & $[v - A\frac{\partial}{\partial\omega}] = [v]$, we may assume $A=0$.

Note $\Theta^\alpha\equiv 0 \mod N$ (i.e. $f^*\Theta^\alpha=0$) $\Rightarrow v\lrcorner(\Theta^\alpha\wedge\Theta^\sigma)\equiv 0 \mod N$.

$\therefore\ v\lrcorner\Theta^\alpha = B^\alpha,\quad v\lrcorner\, d\Theta^\rho \equiv -E^\rho_\alpha\,\omega \mod N$

$v\lrcorner\, d\Theta^\mu \equiv -C^\mu\omega \mod N$

$$\therefore \quad d(v \lrcorner \theta^\rho) + v \lrcorner d\theta^\rho = dB^\rho - E^\rho_\alpha B^\alpha \omega \equiv 0 \pmod N$$

$$d(v \lrcorner \theta^\mu) + v \lrcorner d\theta^\mu = dB^\mu - C^\mu \omega \equiv 0 \pmod N$$

That is, the coefficients of $v \circ f$ along f,

$$B^\alpha(s) = B^\alpha \circ f \quad \& \quad C^\mu(s) = C^\mu \circ f$$

must satisfy the ODE system, where x is coord on N, $\omega = \overset{\circ}{\underset{\#}{D}}(x)\, dx$

$$\frac{dB^\rho}{dx} - E^\rho_\alpha(x) B^\alpha(x) D(x) = 0$$

$$\frac{dB^\mu}{dx} - C^\mu(x) D(x) = 0$$

set of solutions to this system is given by (s_1) arbitrary functions

$$C^\mu(x) \text{ on } N, \quad \mu = s - s_1, \dots, s$$

and arbitrary constants $B^\alpha(a)$, if $N = [a, b]$. (initial conditions)

<u>Cauchy characteristics</u>.

def. The <u>associated system</u> (= space of Cauchy characteristic vector fields) of EDS $\mathcal{I}$ generated by $W^* \subset T^*X$ is

$$\mathfrak{A}(\mathcal{I}) = \{ v \in \mathfrak{X}(X) : v \lrcorner \mathcal{I} \subset \mathcal{I} \}$$

<u>Exercise</u> Prove $u, v \in \mathfrak{A}(\mathcal{I}) \Rightarrow [u,v] \in \mathfrak{A}(\mathcal{I})$. (Use $d\mathcal{I} \subset \mathcal{I}$).

assume $\{ v(x) : v \in \mathfrak{A}(\mathcal{I}) \} \subseteq T_x X$ has constant rank, so $\mathfrak{A}(\mathcal{I})$ defines a subbundle $A(\mathcal{I}) \subset TX$ s.t. $\mathfrak{A}(\mathcal{I}) = C^\infty(X, A(\mathcal{I}))$.

<u>Exercise</u>. $A(\mathcal{I}) \subset W^{*\perp}$ (whose fiber at $x \in X$ is

$$W^{*\perp}_{(x)} = \{ v \in T_x X : \theta(v) = 0 \ \forall\, \theta \in W^*_{(x)} \}.$$

def. The Cauchy characteristic system associated to $\mathcal{I}$ is

$$C(\mathcal{I}) = A(\mathcal{I})^\perp \subset T^*X.$$

(over for Theorem)