14. Strongly Non-degenerate variational problems.

(pp 92-96 with Thm I. e 34 p. 94). Now contained in Elastic curves in $\mathbb{R}^3$ lectures.

When does the momentum space $Y = Z_1$?

We have variational problem $(\mathcal{I}, \omega, \varphi)$ on $Z = X \times \mathbb{R}^s$

$\mathcal{I}$ generated by $W^* = \mathrm{span}\{\theta^1, \ldots, \theta^s\}$. Coords $\lambda_1, \ldots, \lambda_s$ on $\mathbb{R}^s$

Assume $L^* = \mathrm{span}\{\omega, \theta^1, \ldots, \theta^s\}$ is completely integrable,

This $\Rightarrow$ $(\mathcal{I}, \omega)$ is in good form + $\exists$ admissible coframe on X

$\omega, \theta^\rho, \theta^\mu, \pi^\mu$, $\rho = 1, \ldots, s_1$, $\mu = s_1+1, \ldots, s$

so $\dim X = 1 + s + s - s_1$ $\quad s_1 = \dim W^*/W_1^*$, first Cartan integer.

such that the structure equations are

(i) $d\theta^\rho \equiv 0 \mod L^* \wedge L^*$

(ii) $d\theta^\mu \equiv -\pi^\mu \wedge \omega \mod L^* \wedge L^*$.

(iii) $d\omega \equiv 0 \mod L^* \wedge L^*$.

Assume also that

$$d\varphi \equiv A_\mu \pi^\mu \wedge \omega \mod L^* \wedge L^*, \quad A_\mu \in C^\infty(X)$$

(always true if $\varphi = L\omega$, where $L \in C^\infty(X)$ is the Lagrangian).

Then (a) $dA_\mu \wedge \pi^\mu \wedge \omega \equiv 0 \mod T^*X \wedge L^* \wedge L^*$

[Proof. $0 = dd\varphi = dA_\mu \wedge \pi^\mu \wedge \omega + A_\mu \underbrace{d(\pi^\mu \wedge \omega)}_{\substack{\shortparallel \\ -dd\theta^\mu = 0}}$ mod $T^*X \wedge L^* \wedge L^*$, $\because dL^* \subset L^*$ $T^*X \wedge L^* \wedge L^*$ by (ii).

Cartan's Lemming (a) $\Rightarrow$ $dA_\mu \equiv A_{\mu\nu} \pi^\nu \mod L^*$

where $A_{\mu\nu} = A_{\nu\mu} \in C^\infty(X)$.

(no η^i if $(\mathcal{I}, \omega)$ has no Cauchy characteristics).

Pf. Let $\omega, \theta^\alpha, \pi^\mu, \eta^i$ be a coframe on X

$$dA_\mu \equiv A_{\mu j} \eta^j + A_{\mu\nu} \pi^\nu \mod L^*$$

Substitute (a) to get: $(A_{\mu j}\eta^j + A_{\mu\nu}\pi^\nu)\wedge\pi^\mu\wedge\omega \equiv A_{\mu j}\eta^j\wedge\pi^\mu\wedge\omega + A_{\mu\nu}\pi^\nu\wedge\pi^\mu\wedge\omega \equiv 0 \mod T^*X\wedge L^*\wedge L^*$

$\cancel{\mod L^*\wedge L^* \Rightarrow A_{\mu i} = 0 \ \forall \mu, i}$

Contract with $\frac{\partial}{\partial\eta^j} \Rightarrow A_{\mu j}\,\pi^\mu\wedge\omega \equiv 0 \mod L^*\wedge L^*$

$\Rightarrow A_{\mu j} = 0 \quad \forall \mu, i$

Contract with $\frac{\partial}{\partial\pi^\mu}\wedge\frac{\partial}{\partial\pi^\nu} \Rightarrow (A_{\mu\nu} - A_{\nu\mu})\,\omega \equiv 0 \mod L^* \Rightarrow A_{\mu\nu} = A_{\nu\mu}$. //

Def. $(A_{\mu\nu})$ is the <u>quadratic form</u> <u>associated</u> to $(\mathcal{I}, \omega, \varphi)$.

$(\mathcal{I}, \omega, \varphi)$ is <u>strongly</u> <u>nondegenerate</u> if $\det(A_{\mu\nu}) \neq 0$.

<u>Exercise</u> a classical variational problem $(\mathcal{I}, \omega, \varphi)$ on $X = J^1(\mathbb{R}, M)$ is strongly nondegenerate iff $\det(L_{\dot y^\alpha \dot y^\beta}) \neq 0$, where $\varphi = L\omega$.

Theorem I. e. 34 (P.94) If $(\mathcal{I}, \omega, \varphi)$ is strongly nondegenerate,

then 1) $Y = Z_1$

2) $\dim Y = 2s+1$

3) $\psi\wedge\Phi^s \neq 0$ i.e. $(\mathcal{I}, \omega, \varphi)$ is non-degenerate.

<u>Proof</u> $\psi = \varphi + \lambda_\alpha\theta^\alpha$ $\qquad \alpha = 1, \dots, s$

$\Psi = d\psi + d\lambda_\alpha\wedge\theta^\alpha + \lambda_\rho\, d\theta^\rho + \lambda_\mu\, d\theta^\mu$ $\qquad \rho = 1, \dots, s_1$, $\mu = s_1+1, \dots, s$.

on $Z = X\times\mathbb{R}^s$, λ_α coordinates on $\mathbb{R}^s$.

and $\omega, \theta^\rho, \theta^\mu, \pi^\mu$ admissible coframe on X.

By (i), (ii), & (iii),

$d\theta^\rho \equiv a^\rho_\alpha\,\theta^\alpha\wedge\omega \mod S = W^*\wedge W^*$

$d\theta^\mu \equiv -\pi^\mu\wedge\omega + a^\mu_\alpha\,\theta^\alpha\wedge\omega \mod S$

$d\omega \equiv b_\alpha\,\theta^\alpha\wedge\omega \mod S$

$d\varphi \equiv A_\mu\,\pi^\mu\wedge\omega + c_\alpha\,\theta^\alpha\wedge\omega \mod S$

$\therefore \quad \Psi = d\psi = d\varphi + d\lambda_\alpha \wedge \theta^\alpha + \lambda_\rho d\theta^\rho + \lambda_\mu d\theta^\mu$

$\equiv A_\mu \pi^\mu \wedge \omega + c_\alpha \theta^\alpha \wedge \omega + d\lambda_\alpha \wedge \theta^\alpha + \lambda_\rho a^\rho_\alpha \theta^\alpha \wedge \omega + \lambda_\mu(-\pi^\mu \wedge \omega + a^\mu_\alpha \theta^\alpha \wedge \omega) \mod S$

$\equiv (A_\mu - \lambda_\mu)\pi^\mu \wedge \omega + (d\lambda_\alpha + \ell_\alpha \omega) \wedge \theta^\alpha \mod S$

where $\ell_\alpha = -c_\alpha - \lambda_\rho a^\rho_\alpha - \lambda_\mu a^\mu_\alpha \in C^\infty(Z)$

use dual to $\omega, \theta^\alpha, \pi^\mu, d\lambda_\alpha$

$\Rightarrow \quad C(\Psi) = \text{span}\{\theta^\alpha, (A_\mu - \lambda_\mu)\omega, d\lambda_\alpha + \ell_\alpha \omega\}$

$\therefore \quad Z_1 = \{A_\mu - \lambda_\mu = 0\}$, $s - s_1$ equations

On Z_1, using (a), $d\lambda_\mu = dA_\mu \equiv A_{\mu\nu}\pi^\nu \mod L^*$

strongly non-deg means there exist inverse matrix $A^{\mu\nu} = A^{\nu\mu}$

such that $A_{\mu\nu}A^{\nu\tilde{\mu}} = \delta^{\tilde{\mu}}_\mu$, so we can solve for the

π^μ in terms of the $d\lambda_\mu$.

Thus $\omega, \theta^\alpha, d\lambda_\alpha$ is a coframe field on Z_1 & $\dim Z_1 = 1 + 2s$.

$C_1(\Psi) \overset{\text{def}}{=} C(\Psi)_{Z_1} = \text{span}\{\theta^\alpha, d\lambda_\alpha + \ell_\alpha \omega\}$

There exists an integral element $[v]$ at any point $p \in Z_1$.

[Namely, take $v = \frac{\partial}{\partial\omega} - \ell_\alpha \frac{\partial}{\partial\lambda_\alpha}$, then $v \lrcorner \omega \neq 0$ and

$v \lrcorner \theta^\alpha = 0$ & $v \lrcorner (d\lambda_\alpha + \ell_\alpha\omega) = -\ell_\alpha + \ell_\alpha = 0$.]

$\therefore \quad Y = Z_1$. & $\dim Y = 2s + 1$.

Now $\psi_{Z_1} = \varphi + \lambda_\alpha \theta^\alpha$, $\varphi = L\omega$

$\Psi_{Z_1} \equiv (d\lambda_\alpha + \ell_\alpha \omega) \wedge \theta^\alpha \mod S$

$\therefore \quad C(\Psi_{Z_1}) = \text{span}\{\theta^\alpha, d\lambda_\alpha + \ell_\alpha\omega\} = C_1(\Psi)$

$\Psi^s_{Z_1} = s!\,(d\lambda_1 + \ell_1\omega)\wedge\theta^1 \wedge \cdots \wedge (d\lambda_s + \ell_s\omega)\wedge\theta^s$ $\quad (\because S = W^* \wedge W^*)$

$\Rightarrow \psi_{Z_1} \wedge \Psi^s_{Z_1} = s!\,\varphi \wedge d\lambda_1 \wedge \theta^1 \wedge \cdots \wedge d\lambda_s \wedge \theta^s \neq 0$ on Z_1.