Illustration of Noether ...

15 + 16. Elastic curves in $\mathbb{R}^3$: derived flag & momentum space computation.

pp 147–153.

Illustrate construction of momentum space $Y = Z_3$ & use of Noether's Thm

Recall Lecture 1 exposition of Frenet frame $(x, e) : N \subseteq \mathbb{R} \to \mathbb{E}(3)$ along a regular curve $x : N \to \mathbb{R}^3$ parametrized by arclength s

$e = (e_1, e_2, e_3)$ columns, o.n. frame $\forall\, s \in N$.

$$\frac{dx}{ds} = e_1, \quad \frac{de_1}{ds} = \kappa(s)\, e_2, \quad \frac{de_2}{ds} = -\kappa(s)\, e_1 + \tau(s)\, e_3, \quad \frac{de_3}{ds} = -\tau(s)\, e_2$$

<u>Curvature</u> κ & <u>torsion</u> τ determine the curve up to rigid motion.

Curvature & torsion can be arbitrarily specified. $(\omega^i) = \begin{pmatrix} ds \\ 0 \\ 0 \end{pmatrix}$, $(\omega^i_j) = \begin{pmatrix} 0 & -\kappa & 0 \\ +\kappa & 0 & -\tau \\ 0 & +\tau & 0 \end{pmatrix} ds$,

$(\omega^i; \omega^i_j) = (x, e)^{-1} d(x, e)$, pull-back of the Maurer-Cartan form.

Let $X = \mathbb{E}(3) \times \mathbb{R}^2$, where $\mathbb{R}^2$ has coordinates (κ, τ).

Pfaffian EDS $(\mathcal{I}, \omega)$ on X given by

$$\theta^1 = \omega^2 = 0$$
$$\theta^2 = \omega^3 = 0$$
$$\theta^3 = \omega^3_1 = 0$$
$$\theta^4 = \omega^2_1 - \kappa\omega = 0$$
$$\theta^5 = \omega^3_2 - \tau\omega = 0$$
$$\omega = \omega^1 \neq 0$$

where $((\omega^i), (\omega^i_j))$ is Maurer-Cartan form on $\mathbb{E}(3)$.

Value at a point $(x, A) \in \mathbb{E}(3)$ is

$$(x, A)^{-1}(dx, dA) = (A^{-1}dx, A^{-1}dA)$$

Structure equations

$$d\omega^i = -\omega^i_j \wedge \omega^j \qquad i, j, k = 1, 2, 3$$
$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j \qquad \omega^i_j = -\omega^j_i.$$

Integral manifolds of $(\mathcal{I}, \omega)$ are maps $((x, e), \kappa, \tau) : N \to \mathbb{E}(3) \times \mathbb{R}^2$ such that

$$(x, e)^{-1} d(x, e) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \right) \omega$$

For arbitrarily given fns $\kappa, \tau : N \to \mathbb{R}$, $\exists$ an integral manifold $((x, e), \kappa, \tau) : N \to X$ of $(\mathcal{I}, \omega)$, by Cartan-Darboux.

Consider the variational problem

$$\Phi(N) = \frac{1}{2}\int_N \kappa^2 ds \quad , \quad s = \text{arclength parameter so } \omega = ds.$$

Begin with the more general Lagrangian $L: X \to \mathbb{R}$, $L = L(\kappa)$.

Eventually we'll specialize to the case $L(\kappa) = \frac{1}{2}\kappa^2$.

$\varphi = L\omega$.

Structure equations of (θ, ω) on X are:

$$d\omega = d\omega^1 = \underbrace{-\omega^1_2}_{\substack{\parallel \\ \omega^2_1 = \theta^4 + \kappa\omega}} \wedge \underset{\theta^1}{\overset{\parallel}{\omega^2}} \underbrace{-\omega^1_3}_{\substack{\parallel \\ \theta^3}} \wedge \underset{\theta^2}{\overset{\parallel}{\omega^3}} = (\theta^4 + \kappa\omega)\wedge\theta^1 + \theta^3\wedge\theta^2 \equiv -\kappa\,\theta^1\wedge\omega \quad (W^*\wedge W^*)$$

where $W^* \subset T^*X$ is the span of $\{\theta^1, .., \theta^5\}$

$$d\theta^1 = d\omega^2 = -\omega^2_1\wedge\omega^1 - \omega^2_3\wedge\omega^3 = -(\theta^4 + \kappa\omega)\wedge\omega + (\theta^5 + \tau\omega)\wedge\theta^2 \equiv -(\theta^4 + \tau\theta^2)\wedge\omega \quad (W^*\wedge W^*)$$

$$d\theta^2 = d\omega^3 = -\omega^3_1\wedge\omega^1 - \omega^3_2\wedge\omega^2 = -\theta^3\wedge\omega - (\theta^5 + \tau\omega)\wedge\theta^1 \equiv (\tau\theta^1 - \theta^3)\wedge\omega \quad (W^*\wedge W^*)$$

$$d\theta^3 = d\omega^3_1 = -\omega^3_2\wedge\omega^2_1 = -(\theta^5 + \tau\omega)\wedge(\theta^4 + \kappa\omega) = (\tau\theta^4 - \kappa\theta^5)\wedge\omega \quad (W^*\wedge W^*)$$

$$d\theta^4 = d(\omega^2_1 - \kappa\omega) = -\omega^2_3\wedge\omega^3_1 - d\kappa\wedge\omega - \kappa d\omega$$

$$\equiv (\theta^5 + \tau\omega)\wedge\theta^3 - d\kappa\wedge\omega - \kappa(-\kappa\theta^1\wedge\omega) \equiv \underbrace{(-\tau\theta^3 - d\kappa + \kappa^2\theta^1)}_{\substack{\parallel\, \mathrm{def} \\ -\pi^4}}\wedge\omega$$

$$d\theta^5 = d(\omega^3_2 - \tau\omega) = -\omega^3_1\wedge\omega^1_2 - d\tau\wedge\omega - \tau d\omega$$

$$\equiv +\theta^3\wedge(\theta^4 + \kappa\omega) - d\tau\wedge\omega + \kappa\tau\,\theta^1\wedge\omega \equiv \underbrace{(\kappa\theta^3 - d\tau + \kappa\tau\theta^1)}_{\substack{\parallel \\ -\pi^5}}\wedge\omega$$

where $\pi^4 = d\kappa + \tau\theta^3 - \kappa^2\theta^1$

$\pi^5 = d\tau - \kappa\theta^3 - \kappa\tau\theta^1$

$$d\varphi = \underbrace{dL}_{\substack{\parallel \\ L' d\kappa}}\wedge\omega + L\,d\omega = L'(\pi^4 - \tau\theta^3 + \kappa^2\theta^1)\wedge\omega - L\kappa\,\theta^1\wedge\omega + L(\theta^4\wedge\theta^1 + \theta^3\wedge\theta^2)$$

$$= L'\pi^4\wedge\omega + (L'\kappa^2\theta^1 - L'\tau\theta^3 - L\kappa\theta^1)\wedge\omega - L(\theta^1\wedge\theta^4 + \theta^2\wedge\theta^3)$$

Note: $\mathcal{I},\omega$ is in good form (with $\pi^1 = z\theta^2+\theta^4$, π^4, π^5 as shown.
$\pi^2 = \theta^3 - z\theta^1$
$\pi^3 = k\theta^5 - z\theta^4$

First derived system $W_1^* = \ker \delta \subset W^*$, $\delta : W^* \to \Lambda^2 T^*X / W^* \wedge T^*X$ induced by d

is span$\{\theta^1, \theta^2, \theta^3\}$

Cartan integer $s_1 = s_1(\mathcal{I},\omega) = \operatorname{rank}(W^*/W_1^*) = 2$.

Second derived system $W_2^* = \ker \delta_1 \subset W_1^*$, $\delta_1 : W_1^* \to \Lambda^2 T^*X / W_1^* \wedge T^*X$

$d\theta^1 \equiv -\theta^4 \wedge \omega \notin W_1^* \wedge T^*X$
$d\theta^2 \equiv (z\theta^1 - \theta^3)\wedge\omega \in W_1^* \wedge T^*X$
$d\theta^3 \equiv (z\theta^4 - k\theta^5)\wedge\omega \notin W_1^* \wedge T^*X$

is span$\{\theta^2\}$

Third derived system $W_3^* = \ker \delta_2 \subset W_2^*$, $\delta_2 : W_2^* \to \Lambda^2 T^* X / W_2^* \wedge T^*X$
$= (0)$

$d\theta^2 \equiv (z\theta^1 - \theta^3)\wedge\omega \notin W_2^* \wedge T^*X$

<u>derived flag</u> of $(\mathcal{I},\omega)$ is $W^* \supset W_1^* \supset W_2^* \supset W_3^* = (0)$.

The <u>Euler-Lagrange system</u> $(\mathcal{J},\omega)$ on Y for $(\mathcal{I},\omega,\varphi)$.

On $Z = X \times \mathbb{R}^5$, variables $(\lambda_1, .., \lambda_5)$ on $\mathbb{R}^5$.
$\cong W^* \subset T^*X$. Note $\dim Z = \dim X + 5 = 6+2+5 = 13$.
$(X = E(3)\times\mathbb{R}^2)$

$\psi = \varphi + \lambda_\alpha \theta^\alpha \quad (\alpha = 1,..,5)$

$\Psi = d\psi = d\varphi + d\lambda_\alpha \wedge \theta^\alpha + \lambda_\alpha d\theta^\alpha$

Find the Cartan system $C(\Psi) = \{ v \lrcorner \Psi : v \in C^\infty(Z, TZ)\}$.

A frame field on Z is $\frac{\partial}{\partial\omega}, \frac{\partial}{\partial\lambda_\alpha}, \frac{\partial}{\partial\pi^5}, \frac{\partial}{\partial\pi^4}, \frac{\partial}{\partial\theta^\alpha}$.

As observed earlier, we don't need to find $\frac{\partial}{\partial\omega} \lrcorner \Psi$.

$C(\Psi)$ is generated by the Pfaffian equations

$$\frac{\partial}{\partial \lambda_\alpha} \lrcorner \Psi = \theta^\alpha = 0 \qquad (\text{so } C(\Psi) \supset \mathcal{I}, \text{ as usual}).$$

(i) $$\frac{\partial}{\partial \pi^5} \lrcorner \Psi = \underset{\substack{\parallel \\ 0}}{\frac{\partial}{\partial \pi^5} \lrcorner d\varphi} + \lambda_\alpha \frac{\partial}{\partial \pi^5} \lrcorner d\theta^\alpha = -\lambda_5 \omega = 0$$

(ii) $$\frac{\partial}{\partial \pi^4} \lrcorner \Psi = L'\omega - \lambda_4 \omega = (L' - \lambda_4)\omega = 0$$

(iii) $$\frac{\partial}{\partial \theta^5} \lrcorner \Psi = \underset{\substack{\parallel \\ 0}}{\frac{\partial}{\partial \theta^5} \lrcorner d\varphi} + \underset{\substack{\parallel \\ -d\lambda_5}}{\frac{\partial}{\partial \theta^5} \lrcorner (d\lambda_\alpha \wedge \theta^\alpha)} + \underset{\substack{\equiv \\ \lambda_3(-K\omega) \bmod W^*}}{\underbrace{\lambda_\alpha \frac{\partial}{\partial \theta^5} \lrcorner d\theta^\alpha}}$$

$$\equiv -d\lambda_5 - K\lambda_3 \omega = 0$$

(iv) $$\frac{\partial}{\partial \theta^4} \lrcorner \Psi = \underset{\substack{\equiv \\ 0\ (W^*)}}{\frac{\partial}{\partial \theta^4} \lrcorner d\varphi} + \underset{\substack{\parallel \\ -d\lambda_4}}{\frac{\partial}{\partial \theta^4} \lrcorner (d\lambda_\alpha \wedge \theta^\alpha)} + \underset{\substack{\equiv \\ \lambda_3 z\omega - \lambda_1 \omega \quad (W^*)}}{\lambda_\alpha \frac{\partial}{\partial \theta^4} \lrcorner d\theta^\alpha}$$

$$\equiv -d\lambda_4 - (\lambda_1 - \lambda_3 z)\omega = 0$$

(v) $$\frac{\partial}{\partial \theta^3} \lrcorner \Psi = \underset{\substack{\equiv \\ -L'z\omega}}{\frac{\partial}{\partial \theta^3} \lrcorner d\varphi} + \underset{\substack{\parallel \\ -d\lambda_3}}{\frac{\partial}{\partial \theta^3} \lrcorner (d\lambda_\alpha \wedge \theta^\alpha)} + \underset{\substack{\equiv \\ -\lambda_2 \omega}}{\lambda_\alpha \frac{\partial}{\partial \theta^3} \lrcorner d\theta^\alpha}$$

$$\equiv -d\lambda_3 - (\lambda_2 + L'z)\omega = 0$$

(vi) $$\frac{\partial}{\partial \theta^2} \lrcorner \Psi = \underset{\substack{\equiv \\ 0}}{\frac{\partial}{\partial \theta^2} \lrcorner d\varphi} + \underset{\substack{\parallel \\ -d\lambda_2}}{\frac{\partial}{\partial \theta^2} \lrcorner (d\lambda_\alpha \wedge \theta^\alpha)} + \underset{\substack{\equiv \\ -\lambda_1 z\omega}}{\lambda_\alpha \frac{\partial}{\partial \theta^2} \lrcorner d\theta^\alpha} \equiv -d\lambda_2 - \lambda_1 z\omega = 0$$

(vii) $$\frac{\partial}{\partial \theta^1} \lrcorner \Psi = \underset{\substack{\equiv \\ (L'K^2 - LK)\omega}}{\frac{\partial}{\partial \theta^1} \lrcorner d\varphi} + \underset{\substack{\parallel \\ -d\lambda_1}}{\frac{\partial}{\partial \theta^1} \lrcorner (d\lambda_\alpha \wedge \theta^\alpha)} + \underset{\substack{\equiv \\ \lambda_2 z\omega}}{\lambda_\alpha \frac{\partial}{\partial \theta^1} \lrcorner d\theta^\alpha} \equiv -d\lambda_1 + (L'K^2 - LK + \lambda_2 z)\omega = 0$$

Recall that $Z_1 = \{ z \in Z : \exists$ integral element of $(C(\Psi), \omega)$ at $z \}$

Since $\omega \neq 0$, we see by (i) & (ii) that $Z_1 \subset \{ \lambda_5 = 0, \lambda_4 = L' \}$.

In fact, $Z_1 = \{ \lambda_5 = 0, \lambda_4 = L' \}$, since at any point $z \in Z_1$,

$$C(\Psi) = \text{span} \left\{ \begin{array}{l} \theta^\alpha = 0, \\ \text{iii) } d\lambda_5 = -\kappa \lambda_3 \omega, \quad \text{iv) } d\lambda_4 = (\lambda_3 z - \lambda_1)\omega, \quad \text{v) } d\lambda_3 = -(\lambda_2 + L' z)\omega, \\ \text{vi) } d\lambda_2 = -\lambda_1 z \omega, \quad \text{vii) } d\lambda_1 = (L'\kappa^2 - L\kappa + \lambda_2 z)\omega \end{array} \right\}$$

has a solution $[v]$ at which $v \lrcorner \omega \neq 0$.

Note $\dim Z_1 = \dim Z - 2 = 11$.

Now $C_1(\Psi) \overset{\text{def}}{=} C(\Psi)_{Z_1}$ $\quad$ ($= i^* C(\Psi)$, where $i : Z_1 \hookrightarrow Z$ is inclusion)

$$= \text{span} \left\{ \begin{array}{l} \theta^\alpha = 0, \text{ iii) } 0 = -\kappa \lambda_3 \omega, \quad \text{iv) } dL' = (\lambda_3 z - \lambda_1)\omega, \quad \text{v) } d\lambda_3 = -(\lambda_2 + L' z)\omega, \\ \text{vi) } d\lambda_2 = -\lambda_1 z \omega, \quad \text{vii) } d\lambda_1 = (L'\kappa^2 - L\kappa + \lambda_2 z)\omega \end{array} \right\}$$

$\therefore Z_2 = \{ \lambda_5 = 0, \lambda_4 = L', \lambda_3 = 0 \} \subset Z_1 \subset Z$ $\quad$ (assuming $\kappa \neq 0$)

$\dim Z_2 = 10$

and $$C_2(\Psi) = C(\Psi)_{Z_2} = \text{span} \left\{ \begin{array}{l} \theta^\alpha = 0, \quad \text{iv) } dL' = -\lambda_1 \omega, \quad \text{v) } 0 = -(\lambda_2 + L' z)\omega, \\ \text{vi) } d\lambda_2 = -\lambda_1 z \omega, \quad \text{vii) } d\lambda_1 = (L'\kappa^2 - L\kappa + \lambda_2 z)\omega \end{array} \right\}$$

$\therefore Z_3 = \{ \lambda_5 = 0, \lambda_4 = L', \lambda_3 = 0, \lambda_2 = -L' z \} \subset Z_2 \subset Z_1 \subset Z$, and

$\dim Z_3 = 9$.

$$C_3(\Psi) = C(\Psi)_{Z_3} = \text{span} \left\{ \begin{array}{l} \theta^\alpha = 0, \quad \text{iv) } dL' = -\lambda_1 \omega, \quad \text{vi) } d(L' z) = \lambda_1 z \omega, \\ \text{vii) } d\lambda_1 = (L'\kappa^2 - L\kappa + \lambda_2 z)\omega \end{array} \right\}$$

Note: $\Psi_1 = \Psi_{Z_1} = (d\varphi)_{Z_1} + \sum_1^3 d\lambda_i \wedge \theta^i + dL' \wedge \theta^4 + \sum_1^3 \lambda_i d\theta^i + L' d\theta^4$, $\quad C(\Psi_1) \subset C_1(\Psi)$

$\Psi_2 = \Psi_{Z_2} = (d\varphi)_{Z_2} + \sum_1^2 d\lambda_i \wedge \theta^i + dL' \wedge \theta^4 + \sum_1^2 \lambda_i d\theta^i + L' d\theta^4$, $\quad C(\Psi_2) \subset C_2(\Psi)$

$\Psi_3 = \Psi_{Z_3} = (d\varphi)_{Z_3} + d\lambda_1 \wedge \theta^1 + \lambda_1 d\theta^1 - d(L'z) \wedge \theta^2 - L' z\, d\theta^2 + d(L' \theta^4)$, $\quad C(\Psi_3) \subset C_3(\Psi)$

Exercise: verify

<u>Prop.</u> (II.5.23), p. 151. If $L'L'' \neq 0$, then $Y = Z_3$ and the Euler-Lagrange system (I, ω) is non-degenerate on the 9-dim'l momentum space $Y = Z_3$.

<u>Proof.</u> $Z_3 = \{ \lambda_3 = 0 = \lambda_5, \ \lambda_4 = L', \ \lambda_2 = -L'z \}$.

On Z_3 (i.e. pulled back to Z_3),

$$d\lambda_4 = dL' = L'' d\kappa$$

$$d\lambda_2 = -L'dz - zL''d\kappa = -d(L'z)$$

$$\Rightarrow d\lambda_4 \wedge d\lambda_2 = -L''L' \, d\kappa \wedge dz \neq 0 \text{ on } Z_3 \text{ if } L''L' \neq 0,$$

and then a coframe field on Z_3 is and $d\kappa$ & dz can be solved in terms of $d\lambda_2, d\lambda_4$.

$$\omega, \ \theta^\alpha, \ d\lambda_1, \ d\lambda_2, \ d\lambda_4$$

(*) $C_3(\Psi) = \text{span}\{ \theta^\alpha, \ dL' = -\lambda_1 \omega, \ d(L'z) = \lambda_1 z \omega, \ d\lambda_1 = (L'\kappa^2 - L\kappa + \lambda_2 z)\omega \}$.

$\therefore \ v = \frac{\partial}{\partial \omega} + a^1 \frac{\partial}{\partial \lambda_1} + a^2 \frac{\partial}{\partial \lambda_2} + a^3 \frac{\partial}{\partial \lambda_3} \in T_z Z_3$ satisfies $v \lrcorner C_3(\Psi) = 0$

if $a^4 = -\lambda_1$, $a^2 = -\lambda_1 z$, $a^1 = L'\kappa^2 - L\kappa + \lambda_2 z$, so

$V(C_3(\Psi), \omega) \to Z_3$ is surjective. Hence $Y = Z_3$.

To show non-degeneracy, we must show $\psi_Y \wedge \Psi_Y^4 \neq 0$, where

$$\psi_Y = \overset{L\omega}{\overset{\shortparallel}{\varphi}} + \lambda_1 \theta^1 + \lambda_2 \theta^2 + \lambda_4 \theta^4$$

$$\Psi_Y = d\varphi + d\lambda_1 \wedge \theta^1 + d\lambda_2 \wedge \theta^2 + d\lambda_4 \wedge \theta^4 + \lambda_1 \overbrace{d\theta^1}^{\overset{-(\theta^4 + z\theta^2)\wedge\omega}{\equiv}} + \lambda_2 \overbrace{d\theta^2}^{\overset{(z\theta^1 - \theta^3)\wedge\omega}{\equiv}} + \lambda_4 \overbrace{d\theta^4}^{\overset{(-z\theta^3 - d\kappa + \kappa^2\theta^1)\wedge\omega}{\equiv}}$$

<u>[illegible]</u> $\psi_Y \wedge \Psi_Y^4 = -6! \, L \lambda_4 \, \omega \wedge d\lambda_1 \wedge \theta^1 \wedge d\lambda_2 \wedge \theta^2 \wedge d\lambda_4 \wedge \theta^4 \wedge \theta^3 \wedge \theta^5 \neq 0$

Continue after Lecture on Noether's Theorem.

First integrals and solutions

(pp 152–153)

As we'll see from Noether's Theorem, $\mathbb{E}(3)$ generates 6 first integrals for this var. prob. $(\mathcal{I}, \omega, \varphi)$. In effect, these first integrals are hidden in the fact that the Euler-Lagrange EDS $(\mathcal{J}, \omega)$ on $Y = Z_3$, which is $C_3(\Psi)$, involves only $\lambda_1, \lambda_2, \lambda_4, \kappa, + \tau$.

$X = \mathbb{E}(3) \times \underset{(\kappa,\tau)\text{-space}}{\mathbb{R}^2}$, $\quad Z = X \times \underset{(\lambda_1,..,\lambda_5)\text{-space}}{\mathbb{R}^5}$ $\qquad \varphi = L\omega$, $L = L(\kappa)$

$Y = Z_3 = \{\lambda_5 = 0,\ \lambda_3 = 0,\ \lambda_4 = L',\ \lambda_2 = -\tau L'\} \subseteq Z$

$\mathcal{J} = C_3(\Psi)$ is Pfaffian EDS on Y generated by (all forms pulled back to Y)

$\Theta^\alpha = 0$ (this is $\mathcal{I}$) $\quad \alpha = 1, .., 5$.

i) $d\lambda_4 + \lambda_1\omega = 0$

ii) $d\lambda_2 + \tau\lambda_1\omega = 0$

iii) $d\lambda_1 - (\tau\lambda_2 + L'\kappa^2 - L\kappa)\omega = 0$

λ_4 ii) $\Rightarrow$ ii)$'$ $\lambda_4 d\lambda_2 - \lambda_1\lambda_2\omega = 0$

λ_2 i) $+ \lambda_4$ ii) $\Rightarrow \lambda_2 d\lambda_4 + \lambda_4 d\lambda_2 = 0$ i.e. $d(\lambda_2\lambda_4) = 0$ on Y.

Hence $\lambda_2\lambda_4 = c_1$ is constant on any solution curve (N, f) of $(\mathcal{J}, \omega)$ on Y.

$\therefore \lambda_2\lambda_4$ is a <u>first integral</u> of $(\mathcal{J}, \omega)$ on Y.

<u>Case</u> $L(\kappa) = \frac{1}{2}\kappa^2$.

Now $L' = \kappa$, so $\lambda_4 = \kappa$, $\lambda_2 = -\kappa\tau$ on Y

& the first integral is $c_1 = \lambda_2\lambda_4 = -\kappa^2\tau$ is constant on solution curves.

moreover, iii) becomes

iii) $d\lambda_1 - (+\kappa\tau^2 + \kappa^3 - \frac{1}{2}\kappa^3)\omega = d\lambda_1 + \left(\frac{\lambda_2^2}{\lambda_4} - \frac{1}{2}\lambda_4^3\right)\omega = 0$

Plugging $\lambda_2 = \frac{c_1}{\lambda_4}$ into this gives

iii) $d\lambda_1 + \left(\frac{c_1^2}{\lambda_4^3} - \frac{1}{2}\lambda_4^3\right)\omega = 0$

Combine this with i) $d\lambda_4 + \lambda_1\omega = 0$ to get

$$d\left(\lambda_1^2 + \frac{\lambda_4^4}{4} + \frac{c_1^2}{\lambda_4^2}\right) = 2\lambda_1 d\lambda_1 + \lambda_4^3 d\lambda_4 - \frac{2c_1^2}{\lambda_4^3}d\lambda_4$$

$$= \left[2\lambda_1\left(\cancel{\tfrac{1}{2}\lambda_4^3 - \tfrac{c_1^2}{\lambda_4^3}}\right) + \left(\cancel{\lambda_4^3 - \tfrac{2c_1^2}{\lambda_4^3}}\right)(-\lambda_1)\right]\omega = 0$$

on solution curves of $(\mathcal{J}, \omega)$ on Y.

$\therefore\ \lambda_1^2 + \frac{\lambda_4^4}{4} + \frac{c_1^2}{\lambda_4^2} = c_2$ is constant on solution curves

and is thus a first integral of $(\mathcal{J}, \omega)$ on Y.

On Y, with $L(\kappa) = \frac{1}{2}\kappa^2$, we have $\lambda_4 = \kappa$, $\lambda_2 = -\kappa z$,

and $0 = d\lambda_4 + \lambda_1\omega = d\kappa + \lambda_1\omega = (\kappa' + \lambda_1)\omega$ on solution curves,

where $d\kappa = \kappa'\omega$ on a solution curve. $\therefore\ \lambda_1 = -\kappa'$ on a solution curve.

Hence, on a solution curve,

$(\kappa')^2 + \frac{\kappa^4}{4} + \frac{c_1^2}{\kappa^2} = c_2$, which can be solved by separation of variables and a quadrature.

If κ is a solution on $N = (a, b)$, then

$$z = \frac{-c_1}{\kappa^2}$$

and the curve γ in $\mathbb{R}^3$ with this κ + z is a stationary solution curve of $(\mathcal{J}, \omega, \varphi)$ on X.