17. First integrals and Noether's Theorem.

Variational problem $(\mathcal{I}, \omega, \varphi)$ given by functional

$$\Phi(N, f) = \int_N f^*\varphi$$

where $(N, f) \in \mathcal{V}(\mathcal{I}, \omega)$ is an integral manifold of the Pfaffian EDS $(\mathcal{I}, \omega)$ on X. Associated Euler-Lagrange system $(\mathcal{J}, \omega)$, Pfaffian EDS on momentum space $Y \subset Z = W^* \subset T^*X$, where $\mathcal{I}$ is generated by sections of the subbundle $W^* \subset T^*X$.

Remark $\pi : T^*X \to X$ restricts to $\pi : W^* \to X$.

An integral manifold $(N, f) \in \mathcal{V}(\mathcal{J}, \omega)$ is an embedding $f : N \to Y$, such that $f^*\mathcal{J} = 0$, $f^*\omega \neq 0$ Then $\pi \circ f : N \to X$ is an integral manifold of $(\mathcal{I}, \omega)$ that satisfies the Euler-Lagrange equations of $(\mathcal{I}, \omega, \varphi)$, which are $u \lrcorner d(\varphi + \theta) = 0$, $\forall\ u \in C^\infty(X, T^*X)$, for some $\theta \in C^\infty(X, W^*)$.

Def. A first integral of $(\mathcal{I}, \omega, \varphi)$ is a function $V : Y \to \mathbb{R}$ such that for any $(N, f) \in \mathcal{V}(\mathcal{J}, \omega)$ in Y, $V \circ f : N \to \mathbb{R}$ is constant. i.e. $(N, f) \in \mathcal{V}(\mathcal{J}, \omega)$ iff $f^* dV = 0$, i.e.

Exercise 1. V is constant on any $(N, f) \in \mathcal{V}(\mathcal{J}, \omega)$ iff

$$dV \equiv 0 \mod \mathcal{J}$$

2. If a first integral V is the restriction to Y of a func $V : Z \to \mathbb{R}$, then

$$dV \equiv 0 \mod C(\Psi) \Rightarrow f^* dV = 0 .$$

(Recall $\mathcal{J} = C(\Psi)_Y$).

Def. An *infinitesimal symmetry* of the variational problem $(\mathcal{I}, \omega, \varphi)$ is a vector field v on X such that

(i) $\quad \mathcal{L}_v \mathcal{I} \subset \mathcal{I}$ and $\mathcal{L}_v \varphi \equiv 0 \pmod{\mathcal{I}}$.

Exercise.

a). $\mathcal{L}_v \mathcal{I} \subset \mathcal{I} \iff \exp(tv)^* \mathcal{I} = \mathcal{I}$

b). $\mathcal{L}_v \mathcal{I} \subset \mathcal{I}$ + $\mathcal{L}_v \varphi \equiv 0 \pmod{\mathcal{I}} \iff \begin{cases} \exp(tv)^* \mathcal{I} = \mathcal{I} \text{ and} \\ \exp(tv)^* \varphi \equiv \varphi \pmod{\mathcal{I}}. \end{cases}$

Soln: a). Suppose $\mathcal{I}$ generated by θ^α, complete to frame η^μ.

Then $\exp(tv)^* \mathcal{I} = \mathcal{I} \Rightarrow \exp(tv)^* \theta^\alpha = A^\alpha_\rho \theta^\rho$, some fns $A^\alpha_\rho(t,p)$

$\Rightarrow \mathcal{L}_v \theta^\alpha = \frac{d}{dt}\Big|_0 \exp(tv)^* \theta^\alpha = \frac{\partial A^\alpha_\rho}{\partial t}(0,p)\, \theta^\alpha_{(p)} \in \mathcal{I}_{(p)} \quad \forall p \in X.$

$\mathcal{L}_v(d\theta^\alpha) = d\mathcal{L}_v \theta^\alpha \in d\mathcal{I} \subset \mathcal{I}$.

Conversely, $\mathcal{L}_v(\mathcal{I}) \subset \mathcal{I} \Rightarrow$ for $\exp(tv)^* \theta^\alpha = A^\alpha_\rho(t,p)\, \theta^\rho_{(p)} + B^\alpha_\mu(t,p)\, \eta^\mu_{(p)}$

that $\frac{\partial B^\alpha_\mu}{\partial t} = 0$ on X, since $\frac{d}{dt}\Big|_{t_0} \exp(tv)^* \theta^\alpha \in \mathcal{I} \quad \forall t_0$.

$\therefore B^\alpha_\mu = B^\alpha_\mu(p)$ is indep't of t.

But $\theta^\alpha = \exp(0v)^* \theta^\alpha = A^\alpha_\rho(0,p)\, \theta^\rho_{(p)} + B^\alpha_\mu(p)\, \eta^\mu_{(p)} \Rightarrow B^\alpha_\mu(p) = 0 \quad \forall \alpha, \mu, p.$

i.e. $\exp(tv)^* \mathcal{I} = \mathcal{I}$.

b) similar

skip this construction

Construction. A vector field v on X induces a vector field $\tilde{v}$ on $Z = W^* = X \times \mathbb{R}^s$ as follows. (Not the product lift).

Recall $\psi = \pi^* \varphi + \Theta$, where $\pi: W^* \to X$, $(p,\alpha) \mapsto p$ is the projection and Θ is the canonical 1-form on T^*X, restricted to W^*

$\Theta_{(p,\alpha)}: T_{(p,\alpha)} T^*X \to \mathbb{R}$, $\Theta_{(p,\alpha)}(u) = \alpha(d\pi_{(p,\alpha)} u)$, $\forall u \in T_{(p,\alpha)} T^*X$.

continued on last page, which is p. 5.

<u>Noether's Theorem</u> If $v \in C^\infty(X, TX)$ is an infinitesimal symmetry of $(\mathcal{I}, \omega, \varphi)$, and if $W^\alpha \cong X \times \mathbb{R}^s$ is trivialized by the frame field θ^α, $\alpha = 1, \dots, s$, then the function

$$V = v \lrcorner (\varphi + \lambda_\alpha \theta^\alpha) : Y = X \times \mathbb{R}^s \to \mathbb{R}$$

is a first integral of $(\mathcal{I}, \omega, \varphi)$.

<u>Proof.</u> To show: $f^* dV = 0$ for any $(N, f) \in \mathcal{V}(\mathcal{J}, \omega)$ in Y.

Now $dV = d\, v \lrcorner (\varphi + \lambda_\alpha \theta^\alpha) = \mathcal{L}_v(\varphi + \lambda_\alpha \theta^\alpha) - v \lrcorner d(\varphi + \lambda_\alpha \theta^\alpha)$,

$\mathcal{L}_v(\varphi + \lambda_\alpha \theta^\alpha) \in \mathcal{I}$ by definition of inf. sym. v,

& $f^*\big(\underbrace{v \lrcorner d(\varphi + \lambda_\alpha \theta^\alpha)}_{\in C(\Psi)}\big) = 0$ $\forall v \in C^\infty(X, TX)$, $\forall (N, f) \in \mathcal{V}(\mathcal{J}, \omega)$ by Euler-Lagrange equations

(here $f : N \to Y = X \times \mathbb{R}^s$, or any $(N, f) \in \mathcal{V}(\mathcal{J}, \omega)$ that satisfies).

Since $f^* \mathcal{I} = 0$, (always $\mathcal{I} \subset C(\Psi)$), we have

$$f^* dV = 0. \quad //$$

<u>Example</u> Classical variational problem $(\mathcal{I}, \omega, \varphi)$ on

$X = J^1(\mathbb{R}, M)$, where in local coords (x, y^α) on $\mathbb{R} \times M$, $\mathcal{I}$ is the Pfaffian EDS generated by

$x, y^\alpha, \dot{y}^\alpha$ are local coords on X,

$\theta^\alpha = dy^\alpha - \dot{y}^\alpha dx = 0$, $\omega = dx \neq 0$ &

$\varphi = L(x, y, \dot{y})\, dx$, with Lagrangian $L : \mathbb{R} \times M \to \mathbb{R}$

On $Z = X \times \mathbb{R}^m$ with coords $(x, y^\alpha, \dot{y}^\alpha, \lambda_\alpha)$.

$\psi = L\, dx + \lambda_\alpha \theta^\alpha = (L - \lambda_\alpha \dot{y}^\alpha) dx + \lambda_\alpha dy^\alpha = -H dx + \lambda_\alpha dy^\alpha$, <u>"Darboux form"</u>

def. $H = (-L + \lambda_\alpha \dot{y}^\alpha)\big|_Y$ is the Hamiltonian associated to $(\mathcal{I}, \omega, \varphi)$.

In non-degenerate case, $Y = Z_1 \subset Z$.

Since $Z_1 = \{ \lambda_\alpha = L_{\dot{y}^\alpha} \} \subseteq Z$,

$$H = -L + \dot{y}^\alpha L_{\dot{y}^\alpha} \quad \text{on } Y. \quad \text{(is actually a fn on } X\text{)}$$

If $v = A\frac{\partial}{\partial x} + B^\alpha \frac{\partial}{\partial y^\alpha} + C^\alpha \frac{\partial}{\partial \dot{y}^\alpha} \in C^\infty(X, TX)$ is an infinitesimal symmetry of $(\mathcal{I}, \omega, \varphi)$, then

$$(*) \quad V = v \lrcorner (L\,dx + \lambda_\alpha \theta^\alpha) = A(L - \dot{y}^\alpha \lambda_\alpha) + B^\alpha \lambda_\alpha$$
$$= -AH + B^\alpha L_{\dot{y}^\alpha}$$

is a first integral, by Noether's Theorem.

<u>Time shift case</u>. If $L = L(y^\alpha, \dot{y}^\alpha)$ does not depend on x, then $v = -\frac{\partial}{\partial x}$ is an infinitesimal symmetry, because

$$\mathcal{L}_v(\theta^\alpha) = 0 \quad \text{and} \quad \mathcal{L}_v(\varphi) = \mathcal{L}_v(L(y^\alpha, \dot{y}^\alpha)\,dx) = d(\underbrace{v \lrcorner L\,dx}_{-L}) + v \lrcorner \underbrace{d(L\,dx)}_{dL\wedge dx}$$
$$= -dL + dL = 0.$$

Hence by $(*)$, $V = +H$ is a first integral of $(\mathcal{I}, \omega, \varphi)$.

For a mechanical system $L = \frac{1}{2} g_{\alpha\beta}(y)\dot{y}^\alpha \dot{y}^\beta - U(y) = T - U$ this first integral is $H = T + U$, the total energy.

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Construction continued:

Any diffeomorphism $F: X \to X$ induces a bundle map (i.e. fiber preserving & linear on the fiber)

$$\tilde{F}: T^*X \to T^*X, \quad \tilde{F}(p,\alpha) = (F^{-1}(p), F^*\alpha)$$

which satisfies $\tilde{F}^*\theta = \theta$.

In fact,
$$\begin{tikzcd} T^*X \arrow[d, "\pi"'] & T^*X \arrow[l, "\tilde{F}"'] \arrow[d, "\pi"] \\ X \arrow[r, "F"'] & X \end{tikzcd}$$
satisfies $\pi \circ \tilde{F} = F^{-1} \circ \pi$

$(\pi(\tilde{F}(p,\alpha)) = F^{-1}(p) = F^{-1}(\pi(p,\alpha)).)$

so at $(p,\alpha) \in T^*X$ and $u \in T_{(p,\alpha)}T^*X$,

$$(\tilde{F}^*\theta)_{(p,\alpha)}(u) = \theta_{\tilde{F}(p,\alpha)}(d\tilde{F}u) = \theta_{(F^{-1}(p), F^*\alpha)}(d\tilde{F}u)$$
$$= (F^*\alpha)(d\pi\, d\tilde{F}u) = (F^*\alpha)(d(F^{-1})d\pi\, u)$$
$$= \alpha(dF\, d(F^{-1})d\pi\, u) = \alpha(d\pi\, u) = \theta_{(p,\alpha)}(u).$$

In particular, if v is an infinitesimal symmetry of $(\mathcal{I}, \omega, u)$, then $\widetilde{\exp(tv)}$ leaves θ invariant and

$\exp(tv)^*\mathcal{I} = \mathcal{I} \Rightarrow \widetilde{\exp(tv)}\, W^* = W^*$.

Then $\tilde{v}$ is the vector field on $Z = W^*$ generated by the 1-parameter group of transformations $\widetilde{\exp(tv)}: W^* \to W^*$.

The notation is thus: $\exp(t\tilde{v}) = \widetilde{\exp(tv)}$

<u>Exercise</u> Let $\sigma \in A^1(X)$ be a 1-form s.t. $\sigma_{(p)} \in W^*_{(p)}$ $\forall p \in X$. Thus, $\sigma: X \to W^*$, $\sigma(p) = (p, \sigma_{(p)})$ is a section of $\pi: W^* \to X$, so $\pi \circ \sigma = id$.

Prove that if θ is the canonical form on T^*X, then $\sigma^*\theta = \sigma$.

[Soln: $(\sigma^*\theta)_{(p)}(u) = \theta_{(p,\sigma_{(p)})}(d\sigma\, u) = \sigma_{(p)}(\underbrace{d\pi\, d\sigma}_{id}\, u) = \sigma_{(p)}(u)$, $\forall u \Rightarrow \sigma^*\theta = \sigma$.]