18. Cyclic coordinates and Clairault's theorem

Classical variational problem $(\mathcal{I}, \omega, \varphi)$ on $X = J^1(\mathbb{R}, M)$,

local coords (x, y^α) on $\mathbb{R}\times M$, $(x, y^\alpha, \dot y^\alpha)$ on X, $\alpha = 1, \dots, m$.

$\mathcal{I}$: $\theta^\alpha = dy^\alpha - \dot y^\alpha dx$, $\omega = dx \neq 0$. $\Rightarrow d\theta^\alpha = -d\dot y^\alpha \wedge dx$

$\varphi = L(x, y^\alpha, \dot y^\alpha)\, dx$, Lagrangian $L : \mathbb{R}\times M \to \mathbb{R}$

Def. y^β, for some β, does not appear in L, then y^β is called a <u>cyclic coordinate</u>.

<u>Prop.</u> If y^β is a cyclic coordinate, then $\frac{\partial}{\partial y^\beta} = v$ is an infinitesimal symmetry and, by Noether's Theorem,

$V = \frac{\partial}{\partial y^\beta} \lrcorner (\varphi + \lambda_\alpha \theta^\alpha) = \lambda_\beta : Z = X\times\mathbb{R}^m \to \mathbb{R}$ restricted to $Y \subset Z$

is a first integral of $(\mathcal{I}, \omega, \varphi)$. If L is non-degenerate, then $Y = Z_1 = \{\lambda_\alpha = L_{\dot y^\alpha} : \forall\alpha\}$ and so on Y,

$$V = L_{\dot y^\beta}.$$

<u>Proof.</u> $\mathcal{L}_v \theta^\alpha = v \lrcorner d\theta^\alpha + d\, v\lrcorner \theta^\alpha = 0 + d\delta^\alpha_\beta = 0 \quad \forall\alpha$

$\Rightarrow \mathcal{L}_v \mathcal{I} \subset \mathcal{I}$

$\mathcal{L}_v \varphi = v\lrcorner d\varphi + d\, v\lrcorner\varphi = \frac{\partial}{\partial y^\beta} \lrcorner (L_{y^\alpha} dy^\alpha + L_{\dot y^\alpha} d\dot y^\alpha)\wedge dx + d(\overset{0}{\overbrace{\frac{\partial}{\partial y^\beta}\lrcorner\varphi}}) = L_{y^\beta} dx = 0$

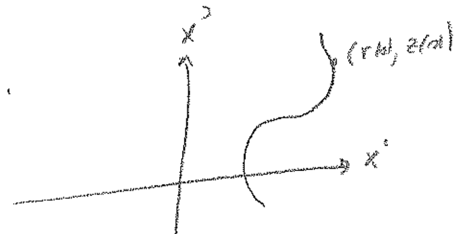
since $L_{y^\beta} = 0$.

$\therefore v$ is an inf. sym. of $(\mathcal{I}, \omega, \varphi)$. //

<u>Example</u>. Surfaces of revolution in $\mathbb{R}^3$.

Given $(r(s), 0, z(s))$, a regular curve in the x^1x^3-plane, $r(s) > 0 \ \forall s$.

Regular means $w(s) = \sqrt{\left(\frac{dr}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} > 0 \quad \forall s$.



Surface of revolution $\vec{x} : M \subset \mathbb{R}^2 \to \mathbb{R}^3$ is

$\vec{x}(s, t) = (r(s)\cos t,\ r(s)\sin t,\ z(s))$ <u>Note</u>: r = distance from $\vec{x}$ to axis of rotation.

$d\vec{x} = (r'\cos t,\ r'\sin t,\ z')\,ds + (-r\sin t,\ r\cos t,\ 0)\,dt$

Induced Metric $d\vec{x} \cdot d\vec{x} = w^2 ds^2 + r^2 dt^2$

$X = J^1(\mathbb{R}, M)$, coords $x, s, t, \dot{s}, \dot{t}$

$\mathcal{I}$: $\theta^1 = ds - \dot{s}\,dx$ $\qquad \omega = dx \neq 0 \qquad d\theta^1 = -d\dot{s} \wedge dx$

$\theta^2 = dt - \dot{t}\,dx \qquad\qquad\qquad\qquad d\theta^2 = -d\dot{t} \wedge dx$

Lagrangian $L(x, s, t) = \frac{1}{2}\left(w(s)^2 \dot{s}\dot{s} + r(s)^2 \dot{t}\dot{t}\right) : \mathbb{R} \times M \to \mathbb{R}$, mechanical system, no potential energy.

Non-degenerate : $\det(L_{\dot{y}^\alpha \dot{y}^\beta}) = \det\begin{pmatrix} L_{\dot{s}\dot{s}} & L_{\dot{s}\dot{t}} \\ L_{\dot{t}\dot{s}} & L_{\dot{t}\dot{t}} \end{pmatrix} = \det\begin{vmatrix} w^2 & 0 \\ 0 & r^2 \end{vmatrix} = (rw)^2 > 0$ on $\mathbb{R} \times M$

$\varphi = L\,dx$. , $\psi = \varphi + \lambda_1\theta^1 + \lambda_2\theta^2$ on $Z = X \times \mathbb{R}^2$.

Classical first order variational problem $(\mathcal{I}, \omega, \varphi)$

Solutions are curves in the surface with stationary energy.

Non-deg $\Rightarrow$ $Y = Z_1 = \{\lambda_1 - L_{\dot{s}} = 0,\ \lambda_2 - L_{\dot{t}} = 0\} \subset Z$

$= \{\lambda_1 = \dot{s}w^2,\ \lambda_2 = \dot{t}r^2\}$

$\Psi = d\psi = dL \wedge dx + d\lambda_1 \wedge \theta^1 + d\lambda_2 \wedge \theta^2 - \lambda_1 d\dot{s} \wedge dx - \lambda_2 d\dot{t} \wedge dx$

$C(\Psi)_Y = \{\theta^1, \theta^2, d\lambda_1 - L_s dx,\ d\lambda_2 - \underset{\substack{\parallel \\ 0}}{L_t} dx\}$ $\Rightarrow t$ is a cyclic variable

$\dot{t}r^2$ is a first integral on Y.

ie. $\dot{t}r^2$ is constant on any stationary curve $(N, \gamma) \in V(\mathcal{I}, \omega)$

Moreover, $L_x = 0 \Rightarrow$ the Hamiltonian H is also a 1st integral.

To find H, express ψ in Darboux form:

$$\psi = L\,dx + \lambda_1(ds - \dot{s}\,dx) + \lambda_2(dt - \dot{t}\,dx)$$

$$= (L - \lambda_1\dot{s} - \lambda_2\dot{t})\,dx + \lambda_1\,ds + \lambda_2\,dt \quad \text{on } Z = X \times \mathbb{R}^2$$

$$\Rightarrow H = (-L + \lambda_1\dot{s} + \lambda_2\dot{t})\Big|_Y = -L + \dot{s}w^2\dot{s} + \dot{t}r^2\dot{t}$$

$$= -\tfrac{1}{2}(w^2\dot{s}^2 + r^2\dot{t}^2) + w^2\dot{s}^2 + r^2\dot{t}^2 \text{ is a 1st integral of } (\mathcal{I}, \omega, \varphi).$$

$$= L$$

In summary, a stationary curve for $(\mathcal{I}, \omega, \varphi)$ is a curve

$$f(x) = \left(x, s(x), t(x), \frac{ds}{dx}, \frac{dt}{dx}, w(s(x))^2\frac{ds}{dx}, r(s(x))^2\frac{dt}{dx}\right) \in Y$$

where $w(s)^2 = \left(\frac{dr}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2$ & $r = r(s(x))$ & $z = z(s(x))$ are given fns defining the profile curve.

satisfying E-L: i) $\frac{d}{dx}\left(w^2\frac{ds}{dx}\right) = L_s = w\frac{dw}{ds}\left(\frac{ds}{dx}\right)^2 + r\frac{dr}{ds}\left(\frac{dt}{dx}\right)^2$ & ii) $r^2\frac{dt}{dx} = \text{constant}$ i.e. $0 = \frac{d}{dx}\left(r^2\frac{dt}{dx}\right)$
$= 2r\frac{dr}{ds}\frac{ds}{dx}\frac{dt}{dx} + r^2\frac{d^2t}{dx^2}$.

$$\gamma(x) = \left(r(s(x))\cos t(x),\ r(s(x))\sin t(x),\ z(s(x))\right)$$

satisfying the Euler-Lagrange equations

$$\frac{d}{dx}L_{\dot{s}} = L_s \mod \mathcal{I} \quad \text{i.e.} \quad f^*\left(\frac{d}{dx}L_{\dot{s}} - L_s\right) = 0$$

$$\frac{d}{dx}L_{\dot{t}} = L_t \quad \text{"} \ \mathcal{I}. \quad f^*\left(\frac{d}{dx}L_{\dot{t}} - L_t\right) = 0.$$

and the 1st integrals $\Rightarrow$

1) $L = \frac{1}{2}\left|\frac{d\gamma}{dx}\right|^2$ is constant along γ

2. $r(x)^2\frac{dt}{dx} = \text{constant}$ along γ.

Remark In general, if $L = \frac{1}{2} g_{\alpha\beta}(y)\dot{y}^\alpha \dot{y}^\beta$ comes from a Riemannian metric on M (coords y^α), then $\Phi(\gamma) = \int_\gamma \frac{1}{2}|\gamma'(x)|^2 dx$ is the energy fcnl on M.

If $\tilde{L} = \sqrt{2L}$, then it gives $\tilde{\Phi}(\gamma) = \int_\gamma ds$, the length functional.

$(I, \omega, \varphi = L\,dx)$ & $(I, \omega, \tilde{\varphi} = \tilde{L}\,dx)$

A solution curve to the Euler-Lagrange eqns of $(I, \omega, L\,dx)$ satisfy $\frac{d}{dx}(L_{\dot{y}^\alpha}) = L_{y^\alpha}$

and $\frac{d}{dx} L = 0$, since $L_x = 0 \Rightarrow L$ is a 1st integral.

Now $\tilde{L}_{y^\alpha} = \dfrac{L_{y^\alpha}}{\sqrt{2L}}$, so along such a curve,

$$\frac{d}{dx}(\tilde{L}_{\dot{y}^\alpha}) = \frac{1}{\sqrt{2L}} \frac{d}{dx}(L_{\dot{y}^\alpha}) = \frac{1}{\sqrt{2L}} L_{y^\alpha} = \tilde{L}_{y^\alpha},$$

which are the Euler-Lagrange eqns of $(I, \omega, \tilde{L}\,dx)$.

Therefore, curves of stationary energy in a Riemannian manifold are geodesics (parametrized proportional to arclength).

Note: $$\Phi(\gamma) = \int_N \gamma^*\varphi = \int_\gamma \overbrace{\tfrac{1}{2} g_{\alpha\beta}(y(x)) \frac{dy^\alpha}{dx}\frac{dy^\beta}{dx}}^{= \frac{1}{2}|\dot{\gamma}(x)|^2} dx$$

$$\tilde{\Phi}(\gamma) = \int_N \gamma^*\tilde{\varphi} = \int_\gamma \underbrace{\left(g_{\alpha\beta}(y(x)) \frac{dy^\alpha}{dx}\frac{dy^\beta}{dx}\right)^{1/2}}_{= |\dot{\gamma}(x)|} dx$$

Exercise Prove Clairault's Theorem: If α is the angle that a geodesic makes with the meridian curve on a surface of revolution, then $r\sin\alpha = \text{constant}$.

[Solution. $\dot{t}r^2$ is a first integral so along any geodesic $\gamma(x)$ in $\vec{X}(s,t) = (r(s)\cos t,\ r(s)\sin t,\ z(s))$ by the integral manifold $f(x) = \left(x, s(x), t(x), \frac{ds}{dx}, \frac{dt}{dx}, w(x)^2\frac{ds}{dx}, r(x)^2\frac{dt}{dx}\right)$

means $\frac{dt}{dx} r(s(x))^2 = c$ is constant.

Recall $d\vec{X} = (r'\cos t, r'\sin t, z')\,ds + r(-\sin t, \cos t, 0)\,dt$

& the meridian curves are the $t=$ constant curves, so

$$\left|\frac{d\gamma}{dx}\right|\sin\alpha = \frac{d\gamma}{dx}\cdot(-\sin t, \cos t, 0) = \left(\left(\cancel{\frac{dr}{dx}\cos t} - r\sin t\frac{dt}{dx}\right)(-\sin t) + \left(\cancel{\frac{dr}{dx}\sin t} + r\cos t\frac{dt}{dx}\right)\cos t\right)$$

$$= \frac{r\frac{dt}{dx}}{\left|\frac{d\gamma}{dx}\right|}$$

and $\left|\frac{d\gamma}{dx}\right| = \sqrt{2L}$ is constant.

Letting x be arclength parameter for γ, we have $\left|\frac{d\gamma}{dx}\right| = 1$

& $\therefore$ $r\sin\alpha = r^2\frac{dt}{dx}$ is constant.]

Exercise 1. Prove that if $r'(s_0) = 0$, & $r_0 = r(s_0)$, then $\gamma(x) = (r_0\cos x,\ r_0\sin x,\ \overset{z(s_0)}{\overset{\shortparallel}{z_0}})$ is a geodesic.

i.e. $f(x) = (x, s_0, x, 0, 1, 0, r_0^2)$ is an int. curve.

2). Implications for geodesics on  or  etc.

use E-L eqns to prove 3). Prove that $\gamma(x) = (r_0\cos x,\ r_0\sin x,\ z_0)$ is a geodesic iff $r'(s_0) = 0$. where $r_0 = r(s_0)$, $z_0 = z(s_0)$.