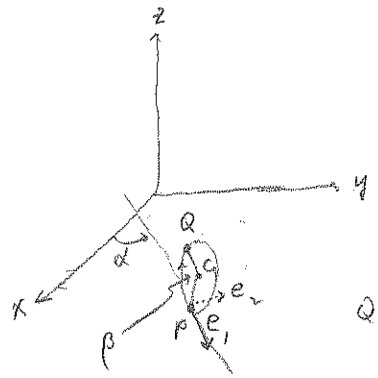
19 + 20 <u>Wheel</u> <u>rolling</u> <u>on</u> <u>a</u> <u>plane</u>.

This example is a first order system, with constraints, that is completely integrable.

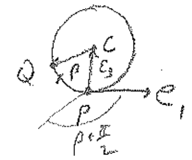
Configuration space: Position of a point Q fixed on the rim of a wheel (= unit circle) free to roll on the x^1x^2-plane without slipping or "twisting".



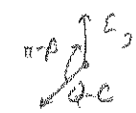
$P = (x, y, 0)$, contact point of wheel on plane.

β = angle from P to Q

e_1 = unit tangent to circle at P in direction of increasing β.



e_1, e_2 oriented o.n. frame in xy-plane



α = the angle from ε_1 to e_1, so $e_1 = (\cos\alpha, \sin\alpha, 0)$

$e_2 = (-\sin\alpha, \cos\alpha, 0)$

$C = (x, y, 1)$ = center of the wheel.

Then $Q - C = -\sin\beta\, e_1 - \cos\beta\, \varepsilon_3$, $\quad \varepsilon_3 = (0,0,1)$.

Proof. $Q - C \in \operatorname{span}\{e_1, \varepsilon_3\}$.

$$(Q-C)\cdot e_1 = \cos\left(\beta + \tfrac{\pi}{2}\right) = -\sin\beta$$

$$(Q-C)\cdot \varepsilon_3 = \cos(\pi - \beta) = -\cos\beta$$

Hence $Q = (x, y, 1) - \sin\beta(\cos\alpha, \sin\alpha, 0) - \cos\beta(0,0,1)$

$= (x - \sin\beta\cos\alpha,\ y - \sin\beta\sin\alpha,\ 1 - \cos\beta)$

for any $(x, y, \alpha, \beta) \in \underbrace{\mathbb{R}^2 \times S^1}_{= \mathcal{F}(\mathbb{R}^2)} \times S^1$, where $S^1 = \mathbb{R}/2\pi$.

so $M = \mathcal{F}(\mathbb{R}^2) \times S^1$

Trajectories of Q in M are the integral curves of the canonical system $(\mathcal{I}, \omega)$ on $J^1(\mathbb{R}, M)$

Local coords are $t, x, y, \alpha, \beta, \dot{x}, \dot{y}, \dot{\alpha}, \dot{\beta}$ (use t on $\mathbb{R}$ here).

The constraint of no slipping or twisting is expressed by the Pfaffian E.D.S. $(\mathcal{K}, \omega)$ on $\mathcal{F}(\mathbb{R}^2)$ = o.n. frame bundle on $\mathbb{R}^2$

$\varphi^1 = \omega^1 - d\beta = 0$, $\omega^1 \neq 0$

$\varphi^2 = \omega^2 = 0$

where (P, e_1, e_2) is an o.n. frame of $\mathbb{R}^2$, as in $\mathcal{F}(\mathbb{R}^2)$

$dP = \omega^1 e_1 + \omega^2 e_2$ so $\omega^1 = dP \cdot e_1 = (dx, dy, 0) \cdot (\cos\alpha, \sin\alpha, 0) = \cos\alpha\, dx + \sin\alpha\, dy$

$\omega^2 = dP \cdot e_2 = -\sin\alpha\, dx + \cos\alpha\, dy$

$de_1 = \omega_1^2 e_2$

$de_2 = \omega_2^1 e_1$, $\omega_1^2 = -\omega_2^1 = de_1 \cdot e_2 = (-\sin\alpha, \cos\alpha, 0)d\alpha \cdot (-\sin\alpha, \cos\alpha, 0) = d\alpha$.

satisfy the structure equations
$$d\omega^1 = -\omega_2^1 \wedge \omega^2$$
$$d\omega^2 = -\omega_1^2 \wedge \omega^1$$
$$d\omega_1^2 = 0$$

The equation $\omega^1 = d\beta$ implies the turning $d\beta$ is rolling by P moving ω^1 in the direction of e_1 (no slipping).

The equation $\omega^2 = 0$ implies no twisting: no infinitesimal motion of P in direction e_2 obtained by twisting wheel on fixed P.

To express the constraint as a submanifold of $J^1(\mathbb{R}, M)$,

$$\left.\begin{matrix}\omega^1 = d\beta \\ \omega^2 = 0\end{matrix}\right\} \Leftrightarrow \left.\begin{cases}\cos\alpha\, dx + \sin\alpha\, dy = d\beta \\ -\sin\alpha\, dx + \cos\alpha\, dy = 0\end{cases}\right\} \Leftrightarrow \begin{matrix} dx = \cos\alpha\, d\beta \\ dy = \sin\alpha\, d\beta\end{matrix} \quad \text{on } J^1(\mathbb{R}, M)$$

$\therefore$ the constraint equations are

(*) $\dot{x} - \cos\alpha\, \dot{\beta} = 0$, two independent equations on $J^1(\mathbb{R}, M)$.
$\dot{y} - \sin\alpha\, \dot{\beta} = 0$ Let X be the submanifold defined by these two equations.

Note $X \subset J^1(\mathbb{R}, M)$ is parametrized by

$$(t, x, y, \alpha, \beta, \dot\alpha, \dot\beta) \mapsto (t, x, y, \alpha, \beta, \cos\alpha\,\dot\beta, \sin\alpha\,\dot\beta, \dot\alpha, \dot\beta)$$

$\dim X = 7.$

The canonical Pfaffian EDS restricted to X is

$$\mathcal{I}: \quad \theta^1 = (dx - \dot x\,dt)|_X = dx - \dot\beta\cos\alpha\,dt$$

$$\theta^2 = (dy - \dot y\,dt)|_X = dy - \dot\beta\sin\alpha\,dt$$

$$\theta^3 = d\alpha - \dot\alpha\,dt$$

$$\theta^4 = d\beta - \dot\beta\,dt$$

$$\omega = dt \neq 0.$$

An integral man. $f: N \to X$ of $(\mathcal{I}, \omega)$ yields an allowable trajectory of Q by $f(t) = (t, x(t), y(t), \alpha(t), \beta(t), \frac{d\alpha}{dt}, \frac{d\beta}{dt})$ satisfying $0 = f^*\theta^1 \Rightarrow \frac{dx}{dt} = \frac{d\beta}{dt}\cos\alpha(t)$ & $0 = f^*\theta^2 \Rightarrow \frac{dy}{dt} = \frac{d\beta}{dt}\sin\alpha(t)$ $\Big\}$ no slipping or twisting.

Note: $\exists$ an allowable trajectory taking any Q_0 to Q_1. Any Q_0 at $(x_0, y_0, \alpha_0, \beta_0) \in M$, Q_1 at $(x_1, y_1, \alpha_1, \beta_1) \in M$, then let $\gamma(t)$ be any smooth unit speed curve in the xy-plane such that $\gamma(0) = (x_0, y_0)$, $\dot\gamma(0) = (\cos\alpha_0, \sin\alpha_0)$, $\gamma(b) = (x_1, y_1)$, $\dot\gamma(b) = (\cos\alpha_1, \sin\alpha_1)$, and $b = \text{length}(\gamma) \equiv \beta_1 - \beta_0 \mod (2\pi)$.

Consider the classical variational problem defined on X by

$$\varphi = L\,dt$$

with Lagrangian

$$L = \tfrac{1}{2}\left|\tfrac{dQ}{dt}\right|^2 = \text{the kinetic energy of the trajectory } Q(t).$$

Now $Q = (x - \sin\beta\cos\alpha,\ y - \sin\beta\sin\alpha,\ 1 - \cos\beta)$, a trajectory

$$\Rightarrow \frac{dQ}{dt} = (\dot x + \sin\beta\sin\alpha\,\dot\alpha - \cos\beta\cos\alpha\,\dot\beta,\ \dot y - \sin\beta\cos\alpha\,\dot\alpha - \cos\beta\sin\alpha\,\dot\beta,\ \sin\beta\,\dot\beta)$$

$$\Rightarrow \left|\frac{dQ}{dt}\right|^2 = \dot x^2 + 2\dot x(\sin\beta\sin\alpha\,\dot\alpha - \cos\beta\cos\alpha\,\dot\beta) + \sin^2\beta\sin^2\alpha\,\dot\alpha^2 \cancel{- 2\sin\alpha\cos\alpha\sin\beta\cos\beta\,\dot\alpha\dot\beta} + \cos^2\beta\cos^2\alpha\,\dot\beta^2$$

$$+ \dot y^2 - 2\dot y(\sin\beta\cos\alpha\,\dot\alpha + \cos\beta\sin\alpha\,\dot\beta) + \sin^2\beta\cos^2\alpha\,\dot\alpha^2 \cancel{+ 2\sin\alpha\cos\alpha\sin\beta\cos\beta\,\dot\alpha\dot\beta} + \cos^2\beta\sin^2\alpha\,\dot\beta^2$$

$$+ \sin^2\beta\,\dot\beta^2$$

$$= \dot x^2 + \dot y^2 + \sin^2\beta\,\dot\alpha^2 + \dot\beta^2 + 2\dot x(\sin\beta\sin\alpha\,\dot\alpha - \cos\beta\cos\alpha\,\dot\beta) - 2\dot y(\sin\beta\cos\alpha\,\dot\alpha + \cos\beta\sin\alpha\,\dot\beta)$$

on $J^1(\mathbb{R}, M)$.

Thus, restricted to X, we get

$$L = \tfrac{1}{2}\Big(\dot\beta^2 + \dot\alpha^2 \sin^2\beta + \dot\beta^2 \quad + 2\dot\beta\cos\alpha\big(\sin\beta \cancel{\sin\alpha}\,\dot\alpha - \cos\beta\cos\alpha\,\dot\beta\big)$$

$$- 2\dot\beta\sin\alpha\big(\cancel{\sin\beta\cos\alpha}\,\dot\alpha + \cos\beta\sin\alpha\,\dot\beta\big)\Big)$$

$$= \tfrac{1}{2}\Big(2\dot\beta^2 + \dot\alpha^2\sin^2\beta \; - 2(\cos\beta\cos^2\alpha + \cos\beta\sin^2\alpha)\dot\beta^2\Big)$$

$$\Rightarrow \quad L = \dot\beta^2(1-\cos\beta) + \tfrac{1}{2}\dot\alpha^2\sin^2\beta \qquad \text{on } X.$$

Consider the variational problem $(\mathcal{I}, \omega, \varphi)$ on X, where $\varphi = L\,dt$, so $\Phi(N,f) = \int_N f^*\varphi$ is the total energy of the trajectory of Q defined by an integral manifold

$$f : N \to X.$$

<u>Euler-Lagrange system for</u> $(\mathcal{I}, \omega, \varphi)$ on $Z = X \times \mathbb{R}^4$

$\psi = \varphi + \lambda_i\theta^i, \quad i = 1,2,3,4.$

[illegible] check this

$$\Psi = d\psi = (L_{\dot\alpha} - \lambda_3)\,d\dot\alpha\wedge dt + (L_{\dot\beta} - \lambda_4 - \lambda_1\cos\alpha - \lambda_2\sin\alpha)\,d\dot\beta\wedge dt$$

$$+ \big(d\lambda_3 - (L_\alpha + \dot\beta(\lambda_2\cos\alpha - \lambda_1\sin\alpha))\,dt\big)\wedge\theta^3 + (d\lambda_4 - L_\beta\,dt)\wedge\theta^4 + \sum_{i=1}^{2} d\lambda_i\wedge\theta^i$$

where

$L_{\dot\alpha} = \dot\alpha\sin^2\beta$

$L_{\dot\beta} = 2\dot\beta(1-\cos\beta)$

$L_\alpha = 0 \quad \Rightarrow \alpha$ is a cyclic variable

$L_\beta = \dot\beta^2\sin\beta + \dot\alpha^2\sin\beta\cos\beta$

Coframe on Z: $dt, \theta^1, \theta^2, \theta^3, \theta^4, d\dot\alpha, d\dot\beta, d\lambda_1, \dots, d\lambda_4$.

Dual frame: $\frac{\partial}{\partial t}, \frac{\partial}{\partial\theta^1}, \frac{\partial}{\partial\theta^2}, \frac{\partial}{\partial\theta^3}, \frac{\partial}{\partial\theta^4}, \frac{\partial}{\partial\dot\alpha}, \frac{\partial}{\partial\dot\beta}, \frac{\partial}{\partial\lambda_1}, \dots, \frac{\partial}{\partial\lambda_4}$

The Euler-Lagrange system is generated by the Pfaffian equations
$u \lrcorner \Psi$, $u \in C^\infty(Z, TZ)$:

i) $\frac{\partial}{\partial \dot\alpha} \lrcorner \Psi = (L_{\dot\alpha} - \lambda_3)\, dt = 0$

ii) $\frac{\partial}{\partial \dot\rho} \lrcorner \Psi = (L_{\dot\rho} - \lambda_4 - \lambda_1 \cos\alpha - \lambda_2 \sin\alpha)\, dt = 0$

(*) $\frac{\partial}{\partial \lambda_i} \lrcorner \Psi = \theta^i = 0$, $i = 1, 2, 3, 4$ $\quad (\, C(\Psi) \supset \mathcal{I}\,)$.

iii) $\frac{\partial}{\partial \theta^3} \lrcorner \Psi = -d\lambda_3 + (L_\alpha - \dot\rho(\lambda_2 \cos\alpha - \lambda_1 \sin\alpha))\, dt = 0$

iv) $\frac{\partial}{\partial \theta^4} \lrcorner \Psi = -d\lambda_4 + L_\rho\, dt = 0$

(v) $\frac{\partial}{\partial \theta^1} \lrcorner \Psi = -d\lambda_1 = 0$

(vi) $\frac{\partial}{\partial \theta^2} \lrcorner \Psi = -d\lambda_2 = 0$

Then (i), (ii), (v), & (vi) $\Rightarrow$ $Z_1 \subset Z$ given by

$$L_{\dot\alpha} = \lambda_3$$

$$L_{\dot\rho} = \lambda_4 + \lambda_1 \cos\alpha + \lambda_2 \sin\alpha$$

$$\lambda_1 = \text{constant}$$

$$\lambda_2 = \text{constant}.$$

Plug these into (iii) & (iv) & use (*) to get (on Z_1)

$$\text{(vii)} \begin{cases} dL_{\dot\alpha} = d\lambda_3 = (L_\alpha - \dot\rho(\lambda_2 \cos\alpha - \lambda_1 \sin\alpha))\, dt \\ dL_{\dot\rho} = d\lambda_4 + (-\lambda_1 \sin\alpha + \lambda_2 \cos\alpha)\, d\alpha = L_\rho\, dt - (\lambda_1 \dot\alpha \sin\alpha - \lambda_2 \dot\alpha \cos\alpha)\, dt \end{cases}$$

To obtain a solution (N, S) of this E-L system, consider the the space $\tilde{X} \subset X$ with coords $t, \alpha, \beta, \dot{\alpha}, \dot{\beta}$ with the classical variational problem $(\tilde{I}, \omega, \tilde{\varphi})$,

$$\tilde{I}: \quad \tilde{\theta}^1 = d\alpha - \dot{\alpha}\,dt = 0$$
$$\tilde{\theta}^2 = d\beta - \dot{\beta}\,dt = 0$$
$$\omega = dt \neq 0$$

$$\tilde{\varphi} = ((1-\cos\beta)\dot{\beta}^2 + \tfrac{1}{2}\sin^2\beta\ \dot{\alpha}^2)\,dt = L(\alpha, \beta, \dot{\alpha}, \dot{\beta})\,dt$$

Prop. (II.a.45, p126) The curves in X obtained from solution curves to $(\tilde{I}, \omega, \tilde{\varphi})$ in $\tilde{X}$ by integrating equations

$$(*) \quad \frac{dx}{dt} = \cos\alpha \frac{d\beta}{dt}$$
$$\frac{dy}{dt} = \sin\alpha \frac{d\beta}{dt}$$

are solution curves of (I, ω, φ) in X. These curves describe energy minimizing trajectories of Q.

Proof. Let $f(t) = (t, \alpha(t), \beta(t), \frac{d\alpha}{dt}, \frac{d\beta}{dt})$ be a solution curve of $(\tilde{I}, \omega, \tilde{\varphi})$ in $\tilde{X}$. Along this curve let

$$\lambda_3 = L_{\dot{\alpha}} \circ f$$
$$\lambda_4 = L_{\dot{\beta}} \circ f$$
$$\lambda_1 = \lambda_2 = 0$$

and $x(t)$, $y(t)$ solutions of $(*)$.

The Euler-Lagrange system of $(\tilde{I}, \omega, \tilde{\varphi})$ is

$$(\circledast)\begin{cases} dL_{\dot{\alpha}} - L_\alpha\,dt = 0 \\ dL_{\dot{\beta}} - L_\beta\,dt = 0 \end{cases}$$

as computed before for any classical var. prob.

Then (next page)

$\tilde{f}(t) = (t, x(t), y(t), \alpha(t), \beta(t), \frac{d\alpha}{dt}, \frac{d\beta}{dt})$ is a curve in X

satisfying:

1). $\tilde{f}^* \mathcal{I} = 0$, $\because$ $\tilde{f}^* \theta^1 = \frac{dx}{dt} - \frac{d\beta}{dt}\cos\alpha(t) = 0$ $\Rightarrow$ (*) holds

$+ \ \tilde{f}^* \theta^2 = \frac{dy}{dt} - \frac{d\beta}{dt}\sin\alpha(t) = 0$

and

2). the Euler-Lagrange system i) - (vi), (*) + (vii).

I feel: i) $L_{\dot\alpha} - \lambda_3 = 0$ by def of λ_3

v) + vi) $\lambda_1 = \lambda_2 = 0$ are constant.

ii) $L_{\dot\beta} - \lambda_4 = 0$ by def of λ_4.

iii) $\tilde{f}^*(d\lambda_3 - L_\alpha dt) = \tilde{f}^*(dL_{\dot\alpha} - L_\alpha dt) = 0$ by (**) or (vii).

iv) $\tilde{f}^*(d\lambda_4 - L_\beta dt) = \tilde{f}^*(dL_{\dot\beta} - L_\beta dt) = 0$ by *.

$\tilde{f}(t)$ yields a stable trajectory of Q. //

Note: Griffiths says not all stable trajectories of Q are solution curves of $(\mathcal{I}, \omega, \varphi)$. More on this later, maybe. //

The variational problem $(\tilde{I}, \omega, \hat{\varphi})$ on $\tilde{X}$. (pp 128-129)