Calculus III

Math 233 — Spring 2007

In-term exam 02/07 — Suggested solutions

This problem set contains sixteen problems numbered 1 through 16. Problems 1 – 15 are multiple choice problems, which each count 5% of your total score. Problem 16 will be hand-graded and counts 25% of your total score.

Problem 1

What is the distance from the point $P(3,-3,4)$ to the x-axis?

A) 0 **B)** 3 **C)** 4 **D)** 5 **E)** $\sqrt{34}$ **F)** None of the above

Points on the x-axis are of the form $(x,0,0)$. The point on the x-axis closest to P will be $Q(3,0,0)$ and the distance between P and Q is

$$\sqrt{(3-3)^2+(-3-0)^2+(4-0)^2}=\sqrt{0^2+(-3)^2+4^2}=5.$$

Problem 2

Which of the following equations describes a plane parallel to $2x-y+4z+4=0$?

A) $x-y+z+2=0$ **B)** $y=2(x+z)$ **C)** $2x^2-y^2+4z^2+4=0$
D) $-x+\frac{1}{2}y-2z=0$ **E)** $2x+y+4z=4$ **F)** None of the above

A plane parallel to the given plane will have normal vector parallel to the normal vector of the given plane. The normal vector of the plane $2x-y+4z+4=0$ is $\langle 2,-1,4\rangle$. We are therefore looking for a plane with normal vector of the form $t\langle 2,-1,4\rangle=\langle 2t,-t,4t\rangle$ where t is a scalar.

Of the given planes, **D)** has normal vector $\langle -1,\frac{1}{2},-2\rangle=-\frac{1}{2}\langle 2,-1,4\rangle$ which is of the required form.

Problem 3

Find a vector parallel to the line described by

$$\frac{x-2}{7} = \frac{y}{3} = \frac{2-z}{5}.$$

A) $\langle 2,0,2\rangle$ **B)** $\langle 7,3,5\rangle$ **C)** $\langle \frac{2}{7}, \frac{1}{3}, \frac{-2}{5}\rangle$ **D)** $\langle -14,-6,10\rangle$ **E)** $\langle 1,0,-1\rangle$
F) None of the above

A line with symmetric equations

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

is parallel to all vectors of the form $t\langle a,b,c\rangle$ where t is a scalar. The vector **D)** can be written $\langle -14,-6,10\rangle = -2\langle 7,3,-5\rangle$ so it is parallel to the given line.

Problem 4

Convert the point given by the cylindrical coordinates $(4, \frac{\pi}{3}, -1)$ to rectangular (Cartesian) coordinates.

A) $(1,\sqrt{3},1)$ **B)** $(\sqrt{2},\sqrt{2},-1)$ **C)** $(\sqrt{3},1,1)$ **D)** $(2,2\sqrt{3},-1)$
E) $(2\sqrt{3},2,-1)$ **F)** None of the above

To convert between cylindrical and rectangular (Cartesian) coordinates, we use the formulas

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z.$$

We therefore find $x = 4\cos\frac{\pi}{3} = 2$, $y = 4\sin\frac{\pi}{3} = 2\sqrt{3}$, and $z = -1$.

Problem 5

What is the scalar projection of $\langle -6, 1, 7\rangle$ onto $\vec{i} + 4\vec{j} - 2\vec{k}$?

A) $\frac{16}{\sqrt{21}}$ **B)** $\frac{16}{\sqrt{86}}$ **C)** $\frac{-16}{\sqrt{21}}$ **D)** $\frac{-16}{\sqrt{86}}$ **E)** $\sqrt{906}$
F) None of the above

The scalar projection of a vector $\vec{b}$ onto $\vec{a}$ is

$$\text{comp}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}.$$

We calculate

$$\text{comp}_{\langle 1,4,-2\rangle}\langle -6,1,7\rangle = \frac{\langle 1,4,-2\rangle\cdot\langle -6,1,7\rangle}{|\langle 1,4,-2\rangle|} = \frac{1\cdot(-6)+4\cdot 1+(-2)\cdot 7}{\sqrt{1^2+4^2+(-2)^2}} = \frac{-16}{\sqrt{21}}.$$

Problem 6

The lines $\langle 1+3t, -1, 4-3t\rangle$ and $\langle 1+t, 1-t, 2\rangle$ intersect in the point $(3, -1, 2)$. What is the angle between the two lines?

A) 0 **B)** $\frac{\pi}{6}$ **C)** $\frac{\pi}{4}$ **D)** $\frac{\pi}{3}$ **E)** $\frac{\pi}{2}$ **F)** None of the above

The line $\langle 1+3t, -1, 4-3t\rangle$ is parallel to the vector $\vec{a} = \langle 3, 0, -3\rangle$, and the line $\langle 1+t, 1-t, 2\rangle$ is parallel to the vector $\vec{b} = \langle 1, -1, 0\rangle$. The angle between $\vec{a}$ and $\vec{b}$ (and therefore between the lines) is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{3\cdot 1+0\cdot(-1)+3\cdot 0}{\sqrt{3^2+0^2+3^2}\sqrt{1^2+(-1)^2+0^2}} = \frac{3}{\sqrt{18\cdot 2}} = \frac{1}{2}.$$

It follows that the angle is $\arccos\frac{1}{2} = \frac{\pi}{3}$.

Problem 7

Write the equation $z = x^2 + y^2$ in spherical coordinates.

A) $\rho^2 = \cos\phi$ **B)** $\rho\sin\phi\tan\phi = 0$ **C)** $\cos^2\phi = \sin^2\phi$
D) $\cos\phi = \rho\sin^2\phi\cos 2\theta$ $\boxed{\textbf{E)}\ \rho\sin^2\phi = \cos\phi}$ **F)** None of the above

To convert between rectangular (Cartesian) and spherical coordinates, we use the formulas

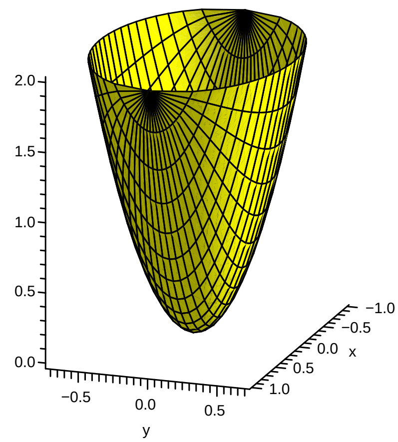
$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi.$$

Inserting into our equation then gives

$$\begin{aligned} z = x^2 + y^2 \qquad &\Leftrightarrow \qquad \rho\cos\phi = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 \\ &\Leftrightarrow \qquad \rho\cos\phi = \rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) \\ &\Leftrightarrow \qquad \cos\phi = \rho\sin^2\phi. \end{aligned}$$

Problem 8

What is the equation of the following surface?



A) $y = \frac{1}{2}x^2 + \frac{1}{4}z^2$ **B)** $z = x^2 - \frac{1}{2}y^2$ **C)** $y = z^2 - \frac{1}{2}x^2$ **D)** $y^2 = x^2 + 2z^2$

E) $z = 2x^2 + 4y^2$ **F)** None of the above

The vertical traces of the surface appear to be parabolas, while the horizontal traces seem to be ellipses. The surface is an elliptic paraboloid, and is described by an equation of the form

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

Problem 9

Which of the following expressions is meaningful?

A) $\vec{a}\cdot\vec{b}+\vec{c}$ **B)** $|\vec{a}|\times(\vec{b}\cdot\vec{c})$ **C)** $(\vec{a}\cdot\vec{b})\times\vec{c}$ **D)** $(\vec{a}+\vec{b})\cdot(\vec{a}\times\vec{c})$ **E)** $\vec{a}\cdot\vec{b}\cdot\vec{c}$
F) None of the above

In expression **A)** we try to sum a scalar ($\vec{a}\cdot\vec{b}$) and a vector, which does not make sense. In expression **B)** we try to take the cross product of two scalars, which does not make sense. In expression **C)** we try to take the cross product of a scalar and a vector, which does not make sense. In expression **D)** we sum two vectors, we take the cross product of two vectors and we take the dot product of two vectors, all operations that are well defined. In expression **E)** we try to take the dot product of three vectors. In whichever order we do this, we end up taking the dot product of a scalar and a vector, which does not make sense.

Problem 10

Which of the following vectors is parallel to the plane described by

$$3(x-2)+(y-7)-4(z+1)=0?$$

A) $\langle 1,1,1\rangle$ **B)** $\langle 1,1,9\rangle$ **C)** $\langle 2,7,-1\rangle$ **D)** $\langle 3,1,-4\rangle$ **E)** $\langle -6,-7,4\rangle$
F) None of the above

If a vector is parallel to a plane, it is orthogonal to the normal vector of the plane. The normal vector of the plane $3(x-2)+(y-7)-4(z+1)=0$ is $\langle 3,1,-4\rangle$. Two vectors are orthogonal only if their dot product is zero. We calculate

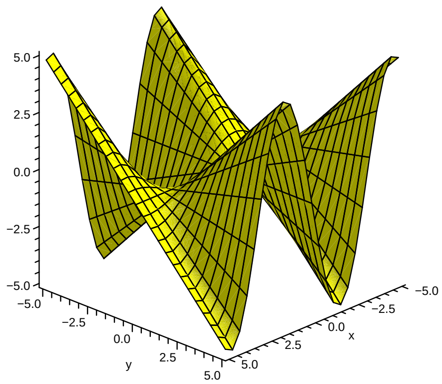
$$\begin{aligned}
\langle 3,1,-4\rangle\cdot\langle 1,1,1\rangle &= 3+1-4=0,\\
\langle 3,1,-4\rangle\cdot\langle 1,1,9\rangle &= 3+1-36=-32,\\
\langle 3,1,-4\rangle\cdot\langle 2,7,-1\rangle &= 6+7+4=17,\\
\langle 3,1,-4\rangle\cdot\langle 3,1,-4\rangle &= 9+1+16=26,\\
\langle 3,1,-4\rangle\cdot\langle -6,-7,4\rangle &= -18-7-16=-41,
\end{aligned}$$

so $\langle 1,1,1\rangle$ is the only vector parallel to the given plane.

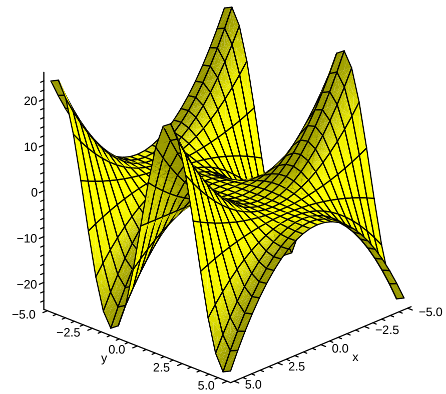
Problem 11

Sketch the graph of the function $f(x,y) = y^2 \cos x$.

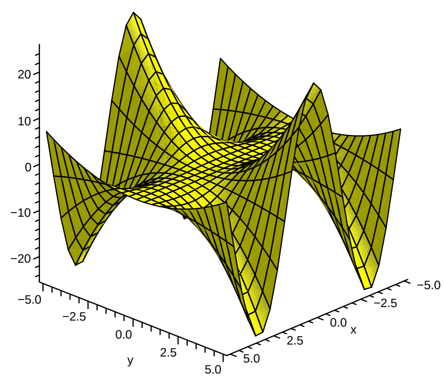
A)



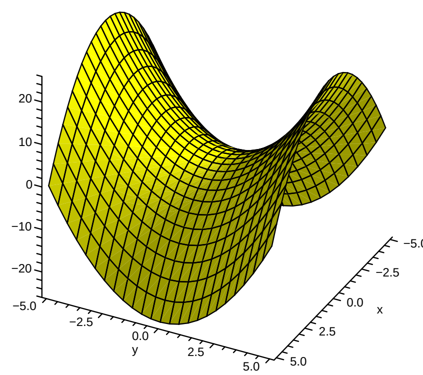
B)



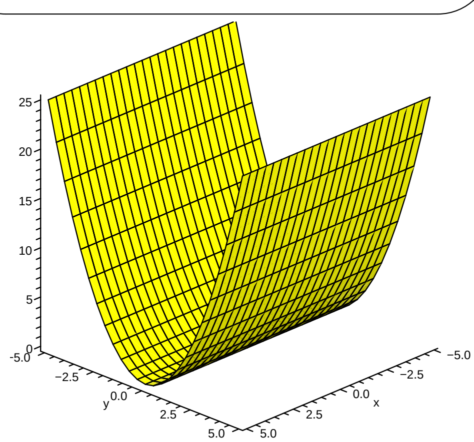
C)



D)



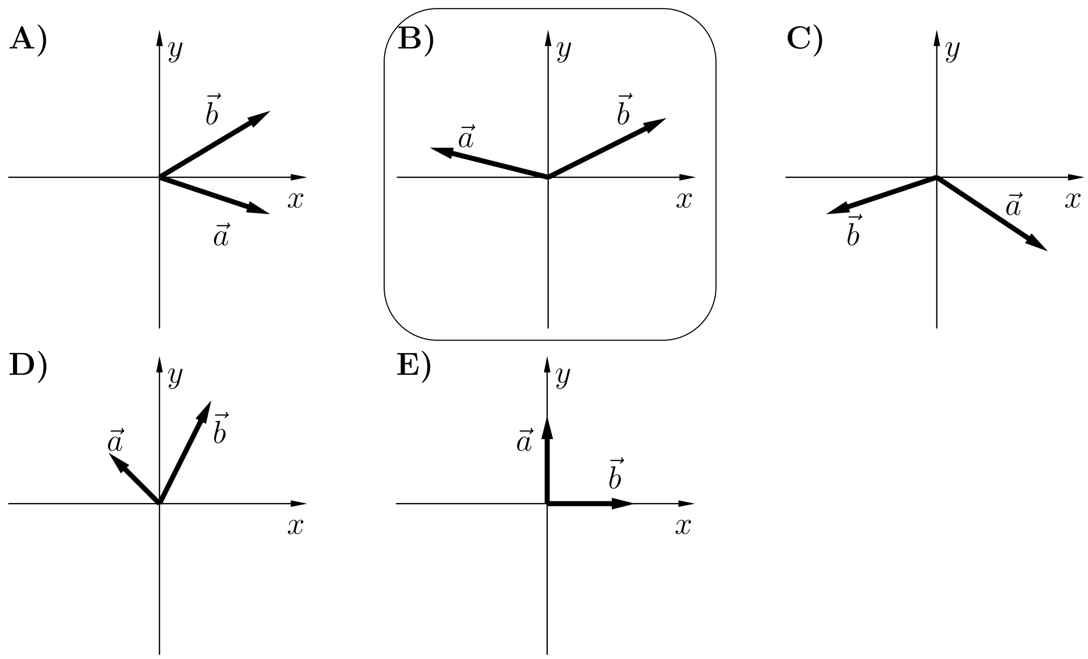
E)



F) None of the above

Problem 12

Which of the following figures depicts two vectors $\vec{a}$ and $\vec{b}$ in the xy-plane such that $\vec{a} \cdot \vec{b} < 0$, $\vec{a} \times \vec{b}$ is parallel to $-\vec{k}$, the x-component of $\vec{a} + \vec{b}$ is 0, and the y-component of $2\vec{a} - \vec{b}$ is 0?



F) None of the above

Problem 13

Which of the following is the equation of a line that lies in both the planes

$$x - 3y + 2z = 2 \qquad \text{and} \qquad x + y = 0?$$

A) $\langle \frac{1}{2} + t, -\frac{1}{2} + t, 0 \rangle$ **B)** $\langle 1 - t, 3 + t, 2 + 2t \rangle$ **C)** $\langle t, -t, 1 - 2t \rangle$
D) $\langle 1 + t, -1 - 3t, -1 + 2t \rangle$ **E)** $\langle 1 + 2t, 1 - 6t, 4t \rangle$ **F)** None of the above

Such a line will be orthogonal to the normal vectors of both planes. The normal vectors of the given planes are $\vec{n}_1 = \langle 1, -3, 2 \rangle$ and $\vec{n}_2 = \langle 1, 1, 0 \rangle$. A vector orthogonal to both $\vec{n}_1$ and $\vec{n}_2$ is parallel to

$$\vec{n}_1 \times \vec{n}_2 = \langle (-3) \cdot 0 - 2 \cdot 1, 2 \cdot 1 - 1 \cdot 0, 1 \cdot 1 - (-3) \cdot 1 \rangle = \langle -2, 2, 4 \rangle.$$

That leaves $\langle 1 - t, 3 + t, 2 + 2t \rangle$ and $\langle t, -t, 1 - 2t \rangle$ as possible alternatives. However, the point $(1, 3, 2)$ is on the line $\langle 1 - t, 3 + t, 2 + 2t \rangle$ but it is not in the plane $x + y = 0$.

Problem 14

Which of the following equations describes a sphere of radius 3?

A) $3x^2 + 3y^2 + 3z^2 = 0$ **B)** $x^2 - y^2 + 9 = z^2$ **C)** $x^2 + y^2 + z^2 - 2x + 4z = 4$
D) $x^2 + y^2 + z^2 - y + 2z = 9$ **E)** $z = x^2 + y^2 - 9$ **F)** None of the above

A sphere of radius 3 will have an equation of the form

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 3^2.$$

Multiplying out, this can also be written as

$$x^2 + y^2 + z^2 - 2x_0x - 2y_0y - 2z_0z = 3^2 - x_0^2 - y_0^2 - z_0^2.$$

Problem 15

What is the angle between the planes $x = y$ and $z = \frac{1}{2}\sqrt{2}(x - y)$?

A) 0 **B)** $\frac{\pi}{6}$ **C)** $\frac{\pi}{4}$ **D)** $\frac{\pi}{3}$ **E)** $\frac{\pi}{2}$ **F)** None of the above

To find the angle between two planes, we calculate the angle between their normal vectors. The normal vectors of the given planes are $\vec{n}_1 = \langle 1, -1, 0 \rangle$ and $\vec{n}_2 = \langle \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}, -1 \rangle$. The angle between these vectors is then given by

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{1 \cdot \frac{1}{2}\sqrt{2} + (-1) \cdot (-\frac{1}{2}\sqrt{2}) + 0 \cdot (-1)}{\sqrt{1^2 + (-1)^2 + 0^2}\sqrt{(\frac{1}{2}\sqrt{2})^2 + (-\frac{1}{2}\sqrt{2})^2 + (-1)^2}} = \frac{\sqrt{2}}{\sqrt{2 \cdot 2}} = \frac{1}{2}\sqrt{2}.$$

It follows that the angle is $\frac{\pi}{4}$ radians.

Name: **Student-ID:**

The following problem will be hand-graded. To earn full credit you need to justify your answers.

Problem 16

a) Let $\vec{a}$ and $\vec{b}$ be two vectors. Show that the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ is the same as the area of the parallelogram spanned by $\vec{a}$ and $\vec{a}+\vec{b}$.

b) Let A_1 denote the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$, and let A_2 denote the area of the parallelogram spanned by $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$. Find the ratio $\frac{A_1}{A_2}$.

a) The area of a parallelogram spanned by two vectors $\vec{u}$ and $\vec{v}$ is given by $A = |\vec{u} \times \vec{v}|$. We have that

$$\vec{a} \times (\vec{a}+\vec{b}) = \vec{a} \times \vec{a} + \vec{a} \times \vec{b} = \vec{a} \times \vec{b},$$

so the areas of the two parallelograms are the same.

b) Using the same formula as above, we find that

$$\begin{aligned} A_2 = |(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})| = |\vec{a} \times \vec{a} + \vec{b} \times \vec{a} - \vec{a} \times \vec{b} - \vec{b} \times \vec{b}| = |\vec{b} \times \vec{a} - \vec{a} \times \vec{b}| \\ = |\vec{b} \times \vec{a} + \vec{b} \times \vec{a}| = 2|\vec{b} \times \vec{a}| = 2A_1. \end{aligned}$$

So the ratio $\frac{A_1}{A_2} = \frac{1}{2}$.