

Calculus III

Math 233 — Spring 2007

Final exam May 3rd. — Suggested solutions

This exam contains twenty problems numbered 1 through 20. All problems are multiple choice problems, and each counts 5% of your total score.

Problem 1

Find the center of the sphere $x^2 + y^2 + z^2 - 6x + 2y + 6 = 0$.

- A)** $(-6, 2, 0)$ **B)** $(-3, 1, 0)$ **C)** $(0, 0, 0)$ **D)** $(1, 1, 1)$ **E)** $(3, -1, 0)$
F) $(3, 1, 3)$ **G)** $(6, -2, 0)$ **H)** $(6, 2, 6)$

Completing squares gives

$$(x - 3)^2 + (y + 1)^2 + z^2 = -6 + 9 + 1 = 4.$$

This is a sphere with center $(3, -1, 0)$ and radius 2.

Problem 2

Compute

$$\text{curl} \langle 5x^2 + 3yz, 7y^2 + 2xz, 3z^2 + 3xy \rangle.$$

- A)** $\langle -3z, -2x, -3y \rangle$ **B)** $\langle -z, 0, x \rangle$ **C)** $\langle 10x, 14y, 6z \rangle$
D) $\langle 3xy - 2xz, 3yz - 3xy, 2xz - 3yz \rangle$ **E)** $\langle x, 0, -z \rangle$ **F)** $\langle 5x^2, 7y^2, 3z^2 \rangle$
G) $\langle 3x, 3y, 2z \rangle$ **H)** $\langle 3yz, 2xz, 3xy \rangle$

Let $\vec{F}(x, y, z) = \langle P, Q, Z \rangle = \langle 5x^2 + 3yz, 7y^2 + 2xz, 3z^2 + 3xy \rangle$. Then the curl of \vec{F} is

$$\begin{aligned} \text{curl } \vec{F} &= \vec{\nabla} \times \vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \\ &= \langle 3x - 2x, 3y - 3y, 2z - 3z \rangle = \langle x, 0, -z \rangle. \end{aligned}$$

Problem 3

Calculate the divergence of $\langle 5x^2 + 3yz, 7y^2 + 2xz, 3z^2 + 3xy \rangle$.

- A) $-2x - 3y - 3z$ B) $x - z$ **C) $10x + 14y + 6z$** D) 0 E) 1
F) $5x^2 + 7y^2 + 3z^2$ G) $3x + 3y + 2z$ H) $3xy + 2xz + 3yz$

We calculate

$$\begin{aligned}\operatorname{div}\langle 5x^2 + 3yz, 7y^2 + 2xz, 3z^2 + 3xy \rangle &= \frac{\partial(5x^2+3yz)}{\partial x} + \frac{\partial(7y^2+2xz)}{\partial y} + \frac{\partial(3z^2+3xy)}{\partial z} \\ &= 10x + 14y + 6z.\end{aligned}$$

Problem 4

Let S be the part of the paraboloid $z = 3 - x^2 - y^2$ above the xy -plane, with upward orientation. Calculate the surface integral (flux integral) of

$$\vec{F}(x, y, z) = \langle x, y, -z \rangle$$

over S .

- A) $\frac{\pi}{2}$ B) $\frac{3\pi}{2}$ C) $\frac{5\pi}{2}$ D) $\frac{7\pi}{2}$ **E) $\frac{9\pi}{2}$** F) $\frac{11\pi}{2}$ G) $\frac{13\pi}{2}$
H) $\frac{15\pi}{2}$

Because S is the graph of a function, $z = g(x, y)$, we can regard x and y as parameters and calculate the surface integral using the formula

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-P \frac{\partial z}{\partial x} - Q \frac{\partial z}{\partial y} + R) dA.$$

This gives

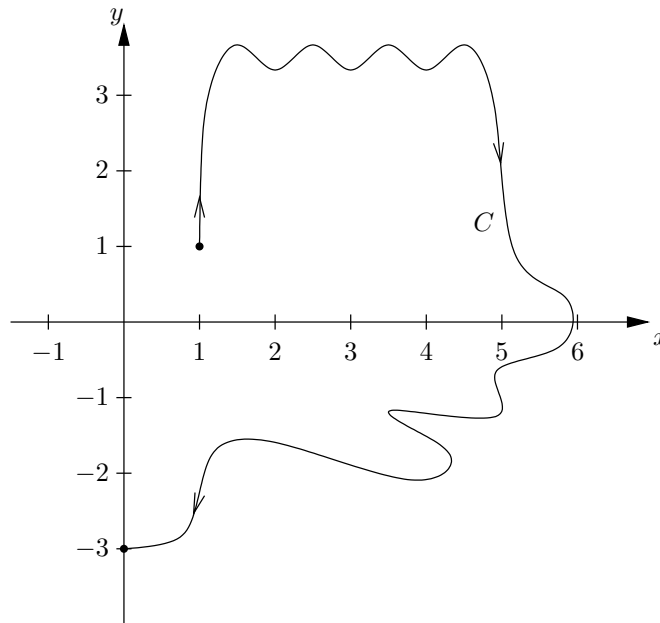
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D [-x(-2x) - y(-2y) - (3 - x^2 - y^2)] dA = \iint_D 3(x^2 + y^2 - 1) dA.$$

The region D is the projection of S to the xy -plane, which is the disk centered at the origin with radius $\sqrt{3}$. Switching to polar coordinates,

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^{\sqrt{3}} 3(r^2 - 1)r dr d\theta = \int_0^{2\pi} 3\left[\frac{1}{4}r^4 - \frac{1}{2}r^2\right]_0^{\sqrt{3}} d\theta = \frac{9}{4} \int_0^{2\pi} d\theta = \frac{9\pi}{2}.$$

Problem 5

Calculate $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = \langle 2xy, x^2 + y \rangle$ and C is the curve in the figure below.



- A) 0 B) -1 C) 2 **D) 3** E) -4 F) -5 G) 6 H) -7

The function $f(x, y) = x^2y + \frac{1}{2}y^2$ satisfies $\vec{\nabla}f = \vec{F}$. The Fundamental Theorem for Line Integrals says that

$$\int_C \vec{F} \cdot d\vec{r} = f(0, -3) - f(1, 1) = \frac{9}{2} - \frac{3}{2} = 3.$$

Problem 6

Let $\vec{a} = \langle 0, 1, -2 \rangle$ and $\vec{b} = \langle -1, 2, -2 \rangle$. Calculate

$$|\vec{a}|, \quad |\vec{b}|, \quad \vec{a} \cdot \vec{b}, \quad |\vec{a} \times \vec{b}|.$$

What is the largest value?

- A) 0 B) 2 C) $\sqrt{5}$ D) $\sqrt{6}$ E) 3 F) 4 G) 5 **H) 6**

We calculate

$$\begin{aligned} |\vec{a}| &= |\langle 0, 1, -2 \rangle| = \sqrt{0^2 + 1^2 + (-2)^2} = \sqrt{5}, \\ |\vec{b}| &= |\langle -1, 2, -2 \rangle| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = 3, \\ \vec{a} \cdot \vec{b} &= \langle 0, 1, -2 \rangle \cdot \langle -1, 2, -2 \rangle = 0 + 2 + 4 = 6, \\ |\vec{a} \times \vec{b}| &= |\langle 0, 1, -2 \rangle \times \langle -1, 2, -2 \rangle| = |\langle 2, 2, 1 \rangle| = 3. \end{aligned}$$

Problem 7

Find the arc length of the curve given by

$$\vec{r}(t) = \langle 2t\sqrt{t}, \cos(3t), \sin(3t) \rangle, \quad 0 \leq t \leq 3.$$

- A) 2 B) 6 C) 10 **D) 14** E) 18 F) 22 G) 26 H) 30

Since $\vec{r}'(t) = \langle 3\sqrt{t}, -3\sin(3t), 3\cos(3t) \rangle$, we get

$$|\vec{r}'(t)| = \sqrt{9t + 9\sin^2 3t + 9\cos^2 3t} = \sqrt{9t + 9} = 3\sqrt{t + 1}.$$

Therefore, the arc length of this curve is

$$\int_0^3 |\vec{r}'(t)| dt = \int_0^3 3\sqrt{t + 1} dt = \left[3 \cdot \frac{2}{3} (t + 1) \sqrt{t + 1} \right]_0^3 = 8\sqrt{4} - 2\sqrt{1} = 14.$$

Problem 8

What kind of surface is described by the parametrization

$$\vec{r}(u, v) = \langle u \cos v, u \sin v, u^2 \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi?$$

- A)** Ellipsoid **B)** Paraboloid **C)** Plane **D)** Cone
E) Hyperboloid of one sheet **F)** Hyperboloid of two sheets **G)** Hemisphere
H) Cylinder

One way to solve this is to observe that $z = x^2 + y^2$, where $x = u \cos v$, $y = u \sin v$, and $z = u^2$. The equation $z = x^2 + y^2$ is that of a paraboloid.

Alternatively, we can look at the grid lines obtained keeping either u or v constant. If u is constant we get circles, while if v is constant the grid lines are parabolas.

Problem 9

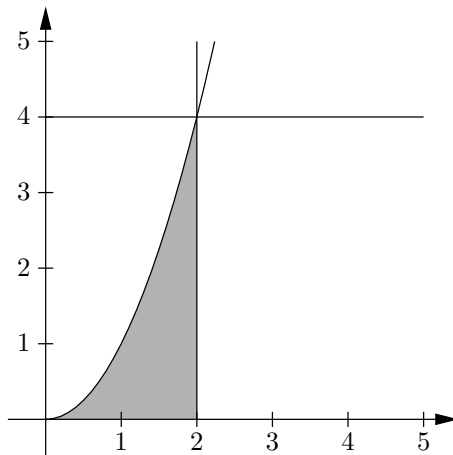
Evaluate

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{1+x^3} \, dx \, dy$$

by reversing the order of integration.

- A) $\frac{25}{9}$ B) $\frac{23}{9}$ C) $\frac{9}{52}$ D) $\frac{469}{52}$ **E) $\frac{52}{9}$** F) $\frac{47}{9}$ G) $\frac{9}{25}$
H) $\frac{226}{25}$

In the picture below, the lines $y = 4$ and $x = 2$ and the parabola $y = x^2$ have been graphed. The integral is taken over the region bounded by the parabola, the line $x = 2$ and the x -axis.



Integrating first with respect to y , we can rewrite the integral as follows:

$$\int_0^2 \int_0^{x^2} \sqrt{1+x^3} \, dy \, dx = \int_0^2 [y\sqrt{1+x^3}]_{y=0}^{y=x^2} \, dx = \int_0^2 x^2 \sqrt{1+x^3} \, dx.$$

The substitution $u = 1 + x^3$ gives $du = 3x^2 \, dx$. Hence

$$\int_0^2 x^2 \sqrt{1+x^3} \, dx = \int_1^9 \frac{1}{3} \sqrt{u} \, du = \frac{1}{3} \left[\frac{2}{3} u \sqrt{u} \right]_1^9 = \frac{2}{9} (27 - 1) = \frac{52}{9}.$$

Problem 10

Let \vec{F} be the vector field

$$\vec{F}(x, y, z) = \langle x, e^{\sin(2z)}, (5 + 3y^{200})^7 \rangle.$$

Let E be the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = -2$ and $z = 4$, while S is the (positively oriented) boundary surface of E .

Use the Divergence Theorem to evaluate

$$\iint_S \vec{F} \cdot d\vec{S}.$$

- A) 2π B) $2\sqrt{2}\pi$ C) 3π D) $2\sqrt{3}\pi$ E) 4π F) $3\sqrt{2}\pi$ G) 5π
H) 6π

By the Divergence Theorem,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV,$$

where E is the solid described above. Here $\operatorname{div} \vec{F} = 1 + 0 + 0 = 1$, so

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E 1 \, dV = \operatorname{volume}(E)$$

Since E is a cylinder with radius 1 and height 6, we know that

$$\operatorname{volume}(E) = \pi \cdot 1^2 \cdot 6 = 6\pi.$$

Problem 11

Which of the following statements are true?

I) The vectors $\langle 1, 3, -4 \rangle$ and $\langle 1, 3, 4 \rangle$ are orthogonal.

II) The vector $\langle 1, 0, 5 \rangle$ is longer than the vector $\langle 3, -3, 1 \rangle$.

III) The cross product of $\langle 1, 0, 1 \rangle$ and $\langle 2, 3, 3 \rangle$ is parallel to $\langle 3, 1, -3 \rangle$.

A) None of them **B)** Only I) **C)** Only II) **D)** Only III)

E) I) and II) **F)** I) and III) **G)** II) and III) **H)** All of them

I) The dot product of the two vectors is nonzero, which implies that they are not orthogonal.

II) $|\langle 1, 0, 5 \rangle| = \sqrt{26}$ and $|\langle 3, -3, 1 \rangle| = \sqrt{19}$, so this statement is correct.

III) $\langle 1, 0, 1 \rangle \times \langle 2, 3, 3 \rangle = \langle -3, -1, 3 \rangle$, so they are indeed parallel.

Problem 12

Find the curvature of the curve

$$\vec{r}(t) = (\sin t)\vec{i} + (\cos t)\vec{j}, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2},$$

at the unique point where the tangent to the curve is parallel to the x -axis.

- A) 0 **B) 1** C) 2 D) 3 E) 4 F) 5 G) 6 H) 7

The short answer is that the curve is a circle of radius 1, and therefore has constant curvature 1.

If we do not make that observation, we can carry out the following calculations. We first find the tangent vector \vec{r}' and the unit tangent vector \vec{T} to the curve,

$$\begin{aligned}\vec{r}'(t) &= \langle \cos t, -\sin t \rangle, \\ |\vec{r}'(t)| &= \sqrt{(\cos t)^2 + (-\sin t)^2} = 1, \\ \vec{T}(t) &= \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \vec{r}'(t) = \langle \cos t, -\sin t \rangle.\end{aligned}$$

The tangent $\vec{r}'(t)$ is parallel to the x -axis when its y -component is zero. That is, when $-\sin t = 0$, which implies $t = 0$. To find the curvature at $t = 0$, we also calculate

$$\vec{T}'(t) = \langle -\sin t, -\cos t \rangle, \quad |\vec{T}'(t)| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1.$$

Thus, the curvature is

$$\kappa(0) = \frac{|\vec{T}'(0)|}{|\vec{r}'(0)|} = 1.$$

Problem 13

Let D be the region in the xy -plane bounded by the upper semicircle $x^2 + y^2 = 1$ and the x -axis, and let C be the positively oriented boundary of D .

Use Green's Theorem to evaluate

$$\int_C \sqrt{x}e^{\cos x} dx + (2x^4y + 4x^2y^3) dy.$$

- A) 0** **B) 4** **C) 8** **D) 12** **E) 16** **F) 20** **G) 24** **H) 28**

Green's Theorem says that

$$\begin{aligned} \int_C \sqrt{x}e^{\cos x} dx + (2x^4y + 4x^2y^3) dy &= \iint_D \left(\frac{\partial(2x^4y + 4x^2y^3)}{\partial x} - \frac{\partial(\sqrt{x}e^{\cos x})}{\partial y} \right) dA \\ &= \iint_D 8x^3y + 8xy^3 dA. \end{aligned}$$

Since D is a circular region, we convert to polar coordinates:

$$\begin{aligned} \iint_D 8x^3y + 8xy^3 dA &= \iint_D 8xy(x^2 + y^2) dA = \int_0^\pi \int_0^1 8(r \cos \theta)(r \sin \theta)(r^2) r dr d\theta \\ &= \int_0^\pi \int_0^1 4r^5(2 \sin \theta \cos \theta) dr d\theta = \int_0^\pi \int_0^1 4r^5(\sin 2\theta) dr d\theta \\ &= \int_0^\pi \left[\frac{2}{3}r^6 \right]_{r=0}^{r=1} \sin 2\theta d\theta = \int_0^\pi \frac{2}{3} \sin 2\theta d\theta = \left[-\frac{1}{3} \cos 2\theta \right]_0^\pi = 0. \end{aligned}$$

Problem 14

The space curve C is parametrized by

$$x = t^3, y = t, z = t^4, \quad 0 \leq t \leq 1.$$

Evaluate

$$\int_C (3x + 8yz) \, ds.$$

- A) $\frac{1}{34}(26\sqrt{26}-1)$ B) $\frac{1}{24}(17\sqrt{17}-1)$ **C) $\frac{1}{18}(26\sqrt{26}-1)$** D) $\frac{1}{13}(23\sqrt{23}-1)$
E) $\frac{1}{15}(17\sqrt{17}-1)$ F) $\frac{1}{16}(28\sqrt{28}-1)$ G) $\frac{1}{19}(15\sqrt{15}-1)$ H) $\frac{1}{13}(29\sqrt{29}-1)$

We first calculate

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \sqrt{(3t^2)^2 + (1)^2 + (4t^3)^2} dt = \sqrt{1 + 9t^4 + 16t^6} dt.$$

Using the parametrization,

$$\int_C (3x + 8yz) \, ds = \int_0^1 (3t^3 + 8t^5) \sqrt{1 + 9t^4 + 16t^6} dt.$$

To integrate this we use the substitution $u = 1 + 9t^4 + 16t^6$, for which

$$\frac{du}{dt} = 36t^3 + 96t^5 = 12(3t^3 + 8t^5).$$

Therefore,

$$\int_C (3x + 8yz) \, ds = \int_1^{26} \frac{1}{12} \sqrt{u} \, du = \frac{1}{12} \left[\frac{2}{3} u \sqrt{u} \right]_1^{26} = \frac{1}{18} (26\sqrt{26} - 1).$$

Problem 15

Find a constant M such that

$$\iint_R xe^y \, dA = M \cdot \text{area}(R),$$

where R is the rectangle

$$R = \{(x, y) : 2 \leq x \leq 4, 0 \leq y \leq 1\}.$$

- A)** $2(e - 1)$ **B)** $2e$ **C)** $3(e - 1)$ **D)** $3e$ **E)** $6(e - 1)$ **F)** $6e$
G) $12(e - 1)$ **H)** $12e$

Note that M will correspond to the average value of the function xe^y over the rectangle R . The area of the rectangle R is $2 \cdot 1 = 2$. Let us compute the integral,

$$\iint_R xe^y \, dA = \int_2^4 \int_0^1 xe^y \, dy \, dx = \int_2^4 [xe^y]_{y=0}^{y=1} \, dx = (e - 1) \left[\frac{1}{2}x^2 \right]_2^4 = 6(e - 1).$$

This implies that $M = 3(e - 1)$.

Problem 16

Use Stokes' Theorem to evaluate

$$\int_C \vec{F} \cdot d\vec{r},$$

where $\vec{F}(x, y, z) = \langle 0, 0, -x^2 \rangle$ and C is the boundary of the part of the plane

$$3x + 2y + z = 6$$

in the first octant. (C is oriented counterclockwise when viewed from above.)

- A) 0 B) 4 **C) 8** D) 12 E) 16 F) 20 G) 24 H) 28

We use Stokes' Theorem to write

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S},$$

where S is the part of the plane $3x + 2y + z = 6$ bounded by C (oriented upward). The curl of \vec{F} is $2x\vec{j}$, so using the formula

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-P \frac{\partial z}{\partial x} - Q \frac{\partial z}{\partial y} + R) dA,$$

we calculate

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \iint_D -2x(-2) dA = \iint_D 4x dA.$$

Here D is the projection of S to the xy -plane. This will be the triangle bounded by the x - and y -axes, and the line $3x + 2y = 6 \Leftrightarrow y = 3 - \frac{3}{2}x$. Thus,

$$\iint_D 4x dA = 4 \int_0^2 \int_0^{3-\frac{3}{2}x} x dy dx = 4 \int_0^2 3x - \frac{3}{2}x^2 dx = 4 \left[\frac{3}{2}x^2 - \frac{1}{2}x^3 \right]_0^2 = 8.$$

Problem 17

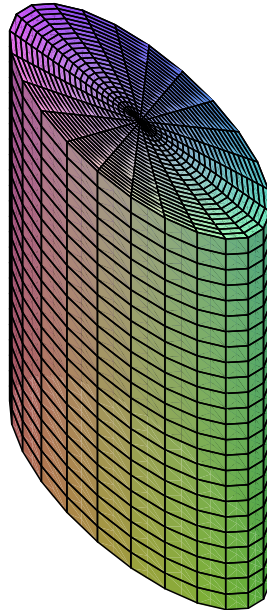
Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 6$, and the two planes $y + z = 7$ and $y + z = 14$.

- A) 6π B) 7π C) 21π D) 36π **E) 42π** F) 63π G) 84π
H) 252π

Using cylindrical coordinates, we need to evaluate the following integral,

$$\begin{aligned} \int_0^{2\pi} \int_0^{\sqrt{6}} \int_{7-y}^{14-y} 1 r \, dz \, dr \, d\theta &= \int_0^{2\pi} \int_0^{\sqrt{6}} ((14-y) - (7-y)) r \, dr \, d\theta = 7 \int_0^{2\pi} \int_0^{\sqrt{6}} r \, dr \, d\theta \\ &= 7 \int_0^{2\pi} \left[\frac{1}{2} r^2 \right]_{r=0}^{r=\sqrt{6}} d\theta = 7 \int_0^{2\pi} 3 \, d\theta = 21 \cdot 2\pi = 42\pi. \end{aligned}$$

Alternatively, we could look at it from a geometrical point of view: If we cut off the bottom (skew) part and place it on top, then we get a circular cylinder with radius $\sqrt{6}$ and height 7. The volume of such a cylinder is $\pi r^2 h = \pi \cdot \sqrt{6}^2 \cdot 7 = 42\pi$.



Problem 18

Which of the vector fields

$$\langle x, y \rangle, \quad \langle y^2 e^x, y^2 e^x \rangle, \quad \langle yz, xz, xy \rangle$$

are conservative?

- A) None of them B) Only $\langle x, y \rangle$ C) Only $\langle y^2 e^x, y^2 e^x \rangle$
D) Only $\langle yz, xz, xy \rangle$ E) $\langle x, y \rangle$ and $\langle y^2 e^x, y^2 e^x \rangle$ **F) $\langle x, y \rangle$ and $\langle yz, xz, xy \rangle$**
G) $\langle y^2 e^x, y^2 e^x \rangle$ and $\langle yz, xz, xy \rangle$ H) All of them

A two-dimensional vector field $\vec{F} = P\vec{i} + Q\vec{j}$ is conservative if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, while a three-dimensional vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is conservative if $\text{curl } \vec{F} = \vec{0}$.

For $\langle x, y \rangle$, we find that

$$\frac{\partial}{\partial y}(x) = 0 = \frac{\partial}{\partial x}(y),$$

so it is conservative.

For $\langle y^2 e^x, y^2 e^x \rangle$,

$$\frac{\partial}{\partial y}(y^2 e^x) = 2ye^x,$$

while

$$\frac{\partial}{\partial x}(y^2 e^x) = y^2 e^x.$$

Therefore, it is not conservative.

Finally,

$$\text{curl}\langle yz, xz, xy \rangle = \langle x - x, y - y, z - z \rangle = \langle 0, 0, 0 \rangle = \vec{0},$$

such that $\langle yz, xz, xy \rangle$ is conservative.

Problem 19

Given the function

$$f(x, y) = 5x^2y + \frac{5}{3}y^3 - 5x^2 - 5y^2 + 7.$$

How many local maxima, local minima, and saddle points does f have?

- A) 1 local minimum, and 1 saddle point
- B) 1 local maximum, and 1 saddle point
- C) 1 local maximum, and 1 local minimum
- D) 1 local maximum, 1 local minimum, and 2 saddle points**
- E) 1 local maximum, 2 local minima, and 1 saddle point
- F) 2 local maxima, 1 local minimum, and 1 saddle point
- G) 2 local minima, and 2 saddle points
- H) 2 local maxima, and 2 saddle points

We first find the partial derivatives of f , $\frac{\partial f}{\partial x} = 10xy - 10x$, $\frac{\partial f}{\partial y} = 5x^2 + 5y^2 - 10y$. The critical points are then where $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. The critical points are $(0, 0)$, $(0, 2)$, $(1, 1)$, $(-1, 1)$.

Next we calculate the second derivatives,

$$\frac{\partial^2 f}{\partial x^2} = 10y - 10, \quad \frac{\partial^2 f}{\partial y^2} = 10y - 10, \quad \frac{\partial^2 f}{\partial x \partial y} = 10x.$$

The discriminant is $D = (10y - 10)^2 - 100x^2$.

At the point $(0, 0)$: $D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0 \Rightarrow$ local maximum,

At the point $(0, 2)$: $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0 \Rightarrow$ local minimum,

At the point $(1, 1)$: $D < 0 \Rightarrow$ saddle point,

At the point $(-1, 1)$: $D < 0 \Rightarrow$ saddle point.

This gives us 1 local maximum, 1 local minimum and 2 saddle points.

Problem 20

Evaluate

$$\iint_S (x + 2y)^2 \, dS,$$

where S is the surface parametrized by

$$x = 2u + e^v, \quad y = -u, \quad z = 2v + 3, \quad 0 \leq u \leq 3, \quad 0 \leq v \leq \frac{1}{2} \ln(2).$$

- A) $21\sqrt{21} - 20\sqrt{20}$ B) $22\sqrt{22} - 20\sqrt{20}$ **C) $22\sqrt{22} - 21\sqrt{21}$**
D) $23\sqrt{23} - 21\sqrt{21}$ E) $23\sqrt{23} - 22\sqrt{22}$ F) $24\sqrt{24} - 22\sqrt{22}$
G) $24\sqrt{24} - 23\sqrt{23}$ H) $25\sqrt{25} - 23\sqrt{23}$

Let \vec{r} be the vector function

$$\vec{r}(u, v) = \langle 2u + e^v, -u, 2v + 3 \rangle, \quad 0 \leq u \leq 3, \quad 0 \leq v \leq \frac{1}{2} \ln(2),$$

which describes S . Then

$$|\vec{r}_u \times \vec{r}_v| = |\langle 2, -1, 0 \rangle \times \langle e^v, 0, 2 \rangle| = |\langle -2, -4, -e^v \rangle| = \sqrt{20 + e^{2v}}.$$

Using the parametrization, $(x + 2y)^2 = (2u + e^v + 2(-u))^2 = e^{2v}$, so

$$\iint_S (x + 2y)^2 \, dS = \int_0^{\frac{1}{2} \ln 2} \int_0^3 e^{2v} \sqrt{20 + e^{2v}} \, du \, dv = 3 \int_0^{\frac{1}{2} \ln 2} e^{2v} \sqrt{20 + e^{2v}} \, dv.$$

To integrate this, we use the substitution $w = 20 + e^{2v}$. Then $\frac{dw}{dv} = 2e^{2v}$, and

$$3 \int_0^{\frac{1}{2} \ln 2} e^{2v} \sqrt{20 + e^{2v}} \, dv = 3 \int_{21}^{22} \frac{1}{2} \sqrt{w} \, dw = \frac{3}{2} \left[\frac{2}{3} w \sqrt{w} \right]_{21}^{22} = 22\sqrt{22} - 21\sqrt{21}.$$