Elementary geometry from an advanced point of view

Math 302 — Fall 2006

Postulates and theorems

Postulate I-0. All lines and planes are sets of points.

Postulate I-1. Given any two different points, there is exactly one line containing them.

Postulate I-2. Given any three different noncollinear points, there is exactly one plane containing them.

Postulate I-3. If two points lie in a plane, then the line containing them lies in the plane.

Postulate I-4. If two planes intersect, then their intersection is a line.

Postulate I-5. Every line contains at least two points. S contains at least three noncollinear points. Every plane contains at least three noncollinear points. And S contains at least four noncoplanar points.

Theorem 1. *Two different lines intersect in at most one point.*

Theorem 2. *If a line intersects a plane not containing it, then the intersection is a single point.*

Theorem 3. *Given a line and a point not on the line, there is exactly one plane containing them.*

Theorem 4. *If two lines intersect, then their union lies in exactly one plane.*

Postulate D-0. d is a function $d\colon S \times S \to \mathbf{R}$.

Postulate D-1. For every $P, Q \in S$, $d(P,Q) \geq 0$.

Postulate D-2. $d(P,Q) = 0$ if and only if $P = Q$.

Postulate D-3. $d(P,Q) = d(Q,P)$ for all $P, Q \in S$

Postulate D-4 (Ruler postulate). Every line has a coordinate system.

Remark. Note that D-4 implies D-1, D-2 and D-3.

Theorem 1. *If f is a coordinate system for L, and $g(P) = -f(P)$ for each $P \in L$, then g is a coordinate system for L.*

Theorem 2. *Let f be a coordinate system for the line L. Let a be any real number and* *for each $P \in L$, let $g(P) = f(P) + a$. Then $g\colon L \to \mathbf{R}$ is a coordinate system for L.*

Theorem 3 (The ruler placement theorem). *Let L be a line, and let P and Q be any two points of L. Then L has a coordinate system in which the coordinate of P is 0 and the coordinate of Q is positive.*

Theorem B-1. *If A-B-C, then C-B-A.*

Theorem B-2. *Of any three points on a line, exactly one is between the other two.*

Theorem B-3. *Any four points of a line can be named in an order A, B, C, D in such a way that A-B-C-D.*

Theorem B-4. *If A and B are any two points, then (1) there is a point C such that A-B-C, and (2) there is a point D such that A-D-B.*

Theorem B-5. *If A-B-C, then A, B and C are three different points on the same line.*

Theorem 1. *If A and B are any two points, then $\overline{AB} = \overline{BA}$.*

Theorem 2. *If C is a point of $\overrightarrow{AB}$ other than A, then $\overrightarrow{AB} = \overrightarrow{AC}$.*

Theorem 3. *If B' and C' are points of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ other than A, then $\angle BAC = \angle B'AC'$.*

Theorem 4. *If $\overline{AB} = \overline{CD}$, then the points A, B are the same as the points C, D in some order.*

Theorem 5. *If $\triangle ABC = \triangle DEF$, then the points A, B, C are the same as the points D, E, F in some order.*

Theorem C-1. *For segments, congruence is an equivalence relation.*

Theorem C-2 (The segment-construction theorem). *Given a segment $\overline{AB}$ and a ray $\overrightarrow{CD}$. There is exactly one point E of $\overrightarrow{CD}$ such that $\overline{AB} \cong \overline{CD}$.*

Theorem C-3 (The segment-addition theorem). *If A-B-C, A'-B'-C', $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.*

Theorem C-4 (The segment-subtraction theorem). *If A-B-C, A'-B'-C', $\overline{AB} \cong \overline{A'B'}$ and $\overline{AC} \cong \overline{A'C'}$, then $\overline{BC} \cong \overline{B'C'}$.*

Theorem C-5. *Every segment has exactly one midpoint.*

Postulate PS-1 (The plane-separation postulate). Given a line and a plane containing it, the set of all points of the plane that do not lie on the line is the union of two disjoint sets such that

(1) each of the sets is convex, and

(2) if P belongs to one of the sets and Q belongs to the other, then the segment $\overline{PQ}$ intersects the line.

Theorem (The postulate of Pasch). *Given a triangle $\triangle ABC$, and a line L in the same plane. If L contains a point P, between A and C, then L intersects at least one of $\overline{AB}$ and $\overline{BC}$.*

Theorem 1. *If P and Q are on opposite sides of the line L, and Q and T are on opposite* *sides of L, then P and T are on the same side of L.*

Theorem 2. *If P and Q are on opposite sides of the line L, and Q and T are on the same side of L, then P and T are on opposite sides of L.*

Theorem 3. *Given a line, and a ray which has its endpoint on the line, but does not lie on the line. Then all points of the ray, except for the endpoint, are on the same side of the line.*

Theorem 4. *Let L be a line, let A and B be points with $A \in L$ and $B \notin L$. Then all points of $\overline{AB} \setminus \{A\}$ lie on the same side of L.*

Theorem 5. *Every side of a triangle lies, except for its endpoints, in the interior of the opposite angle.*

Theorem 6. *If F is in the interior of $\angle BAC$, then $\overrightarrow{AF} \setminus \{A\}$ lies in the interior of $\angle BAC$.*

Theorem. *If A and B are convex, then so is also $A \cap B$.*

Theorem. *If G is any collection of convex sets G_i, then $\bigcap_{G_i \in G} G_i$ is convex.*

Theorem. *If A is any set of points, then the convex hull of A is convex.*

Theorem 8. *The interior of a triangle is always a convex set.*

Theorem 9. *The interior of a triangle is the intersection of the interiors of its angles.*

Theorem 1. *Let L be a line, let A and F be two points of L and let B and G be points on opposite sides of L. Then $\overline{FB}$ does not intersect $\overrightarrow{AG}$.*

Theorem 2. *In $\triangle FBC$, let A be a point between F and C, and let D be a point such that D and B are on the same side of $\overleftrightarrow{FC}$. Then $\overrightarrow{AD}$ intersects at least one of $\overline{FB}$ and $\overline{BC}$.*

Theorem 3 (The crossbar theorem). *If D is in the interior of $\angle BAC$, then $\overrightarrow{AD}$ intersects $\overline{BC}$, in a point between B and C.*

Theorem 1. *The diagonals of a convex quadrilateral always intersect each other.*

Theorem. *If the diagonals of a quadrilateral intersect each other, then the quadrilateral is convex.*

Postulate SS-1 (The space-separation postulate). Given a plane in space. The set of all points that do not lie in the plane is the union of two disjoint sets such that

(1) each of the sets is convex, and

(2) if P belongs to one of the sets and Q belongs to the other, then the segment $\overline{PQ}$ intersects the plane.