Elementary geometry from an advanced point of view

Math 302 — Fall 2008

Postulates and theorems

Postulate (Plane Incidence) (p. 44, 08/27). Undefined terms: plane, line. a) The plane is a set of points, a line is a set of points contained in the plane. b) Two distinct points in the plane are contained in exactly one line. c) There exist three points in the plane, not all on the same line.

Theorem 2.1 (p. 45, 08/27). *Two distinct lines intersect in at most one point.*

Postulate (Ruler) (p. 57, 09/03). Undefined term: d. $d\colon \mathbf{P}\times\mathbf{P}\to\mathbf{R}$, $d\colon (P,Q)\mapsto PQ$ is a mapping such that for each line L there exists a one-to-one correspondence $f\colon L\to\mathbf{R}$, $f\colon P\mapsto f(P)$ where $PQ=|f(Q)-f(P)|$.

Theorem 3.3.1 (p. 58, 09/03). *If f is a coordinate system for L, and $g(P)=-f(P)$ for each $P\in L$, then g is a coordinate system for L.*

Theorem 3.3.2 (p. 59, 09/03). *Let f be a coordinate system for the line L. Let a be any real number and for each $P\in L$, let $g(P)=f(P)+a$. Then $g\colon L\to\mathbf{R}$ is a coordinate system for L.*

Theorem 3.3.3 (Ruler placement) (p. 59, 09/03). *Let L be a line, and let P and Q be any two points of L. Then L has a coordinate system in which the coordinate of P is 0 and the coordinate of Q is positive.*

Theorem B-1 (p. 60, 09/08). *If A-B-C, then C-B-A.*

Theorem B-2 (p. 61, 09/10). *Of any three collinear points, exactly one is between the other two.*

Theorem B-3 (p. 62, 09/10). *Any four collinear points can be named A, B, C, D such that A-B-C-D.*

Theorem B-4 (p. 63, 09/10). *If A and B are any two points, then (1) there is a point C such that A-B-C, and (2) there is a point D such that A-D-B.*

Theorem B-5 (p. 63, 09/10). *If A-B-C, then A, B and C are three different points on the same line.*

Theorem 3.5.1 (p. 66, 09/12). *If A and B are any two points, then $\overline{AB}=\overline{BA}$.*

Theorem 3.5.2 (p. 66, 09/12). *If C is a point of $\overrightarrow{AB}$ other than A, then $\overrightarrow{AB}=\overrightarrow{AC}$.*

Theorem 3.5.3 (p. 66, 09/12). *If B' and C' are points of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ other than A, then $\angle BAC=\angle B'AC'$.*

Theorem 3.5.4 (p. 66, 09/12). *If $\overline{AB}=\overline{CD}$, then the points A,B are the same as the points C,D in some order.*

Theorem 3.5.5 (p. 66, 09/12). *If $\triangle ABC=\triangle DEF$, then the points A,B,C are the same as the points D,E,F in some order.*

Theorem C-1 (p. 69, 09/12). *For segments, congruence is an equivalence relation.*

Theorem C-2 (Segment-construction) (p. 69, 09/15). *Given a segment $\overline{AB}$ and a ray $\overrightarrow{CD}$. There is exactly one point E of $\overrightarrow{CD}$ such that $\overline{AB}\cong\overline{CD}$.*

Theorem C-3 (Segment-addition) (p. 70, 09/15). *If A-B-C, A'-B'-C', $\overline{AB}\cong\overline{A'B'}$ and $\overline{BC}\cong\overline{B'C'}$, then $\overline{AC}\cong\overline{A'C'}$.*

Theorem C-4 (Segment-subtraction) (p. 70, 09/15). *If A-B-C, A'-B'-C', $\overline{AB}\cong\overline{A'B'}$ and $\overline{AC}\cong\overline{A'C'}$, then $\overline{BC}\cong\overline{B'C'}$.*

Theorem C-5 (p. 70, 09/15). *Every segment has exactly one midpoint.*

Postulate (Plane-Separation) (p. 74, 09/17). For every line L there exist two disjoint sets H_1 and H_2 such that a) $\mathbf{P}\setminus L=H_1\cup H_2$; b) H_1 and H_2 are convex; c) for any $P\in H_1$, $Q\in H_2$ we have $\overline{PQ}\cap L\neq\varnothing$.

Theorem (Pasch's Postulate) (p. 74, 09/17). *If a line intersects a triangle, then the line intersects at least two sides of the triangle.*

Theorem 4.2.1 (p. 76, 09/17). *If P and Q are on opposite sides of the line L, and Q and T are on opposite sides of L, then P and T are on the same side of L.*

Theorem 4.2.2 (p. 76, 09/17). *If P and Q are on opposite sides of the line L, and Q and T are on the same side of L, then P and T are on opposite sides of L.*

Theorem 4.2.3 (p. 76, 09/1). *Given a line, and a ray which has its endpoint on the line, but does not lie on the line. Then all points of the ray, except for the endpoint, are on the same side of the line.*

Theorem 4.2.4 (p. 77, 09/22). *Let L be a line, let A and B be points with $A \in L$ and $B \notin L$. Then all points of $\overline{AB} \setminus \{A\}$ lie on the same side of L.*

Theorem 4.2.5 (p. 78, 09/22). *Every side of a triangle lies, except for its endpoints, in the interior of the opposite angle.*

Theorem 4.2.6 (p. 78, 09/22). *If F is in the interior of $\angle BAC$, then $\overrightarrow{AF} \setminus \{A\}$ lies in the interior of $\angle BAC$.*

Theorem 4.2.7 (p. 79, 09/24). *Let $\triangle ABC$ be a triangle, and let F, D and G be points such that B-F-C, A-C-D and A-F-G. Then G is in the interior of $\angle BCD$.*

Theorem 4.3.1 (p. 81, 09/24). *Let L be a line, let A and F be two points of L and let B and G be points on opposite sides of L. Then $\overline{FB}$ does not intersect $\overrightarrow{AG}$.*

Theorem 4.3.2 (p. 81, 09/24). *In $\triangle FBC$, let A be a point between F and C, and let D be a point such that D and B are on the same side of $\overleftrightarrow{FC}$. Then $\overrightarrow{AD}$ intersects at least one of $\overline{FB}$ and $\overline{BC}$.*

Theorem 4.3.3 (Crossbar) (p. 82, 09/24). *If D is in the interior of $\angle BAC$, then $\overrightarrow{AD}$ intersects $\overline{BC}$, in a point between B and C.*

Theorem 4.4.1 (p. 84, 09/26). *$\square ABCD$ is a convex quadrilateral if and only if the diagonals $\overline{AC}$ and $\overline{BD}$ intersect.*

Postulate (Protractor) (p. 95, 09/29). Undefined term: m. m is a function from the set of all angles into $\{x \in \mathbf{R} \colon 0 < x < 180\}$ such that a) if $\overrightarrow{AB}$ is a ray on the edge of a half plane H, then for every number r between 0 and 180, there is exactly one ray $\overrightarrow{AP}$, with P in H and $m\angle PAB = r$; b) if D is in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.

Theorem 5.1 (p. 97, 09/29). *For angles, congruence is an equivalence relation.*

Theorem 5.2 (Angle-construction) (p. 97, 09/29). *Let $\angle ABC$ be an angle, let $\overrightarrow{B'C'}$ be a ray and let H be a half plane whose edge contains $\overrightarrow{B'C'}$. Then there is exactly one ray $\overrightarrow{B'A'}$ with A' in H such that $\angle ABC \cong \angle A'B'C'$.*

Theorem 5.3 (Angle-addition) (p. 97, 09/29). *If D is in the interior of $\angle BAC$, D' is in the interior of $\angle B'A'C'$, $\angle BAD \cong \angle B'A'D'$ and $\angle DAC \cong \angle D'A'C'$, then $\angle BAC \cong \angle B'A'C'$.*

Theorem 5.4 (Angle-subtraction) (p. 98, 09/29). *If D is in the interior of $\angle BAC$, D' is in the interior of $\angle B'A'C'$, $\angle BAD \cong \angle B'A'D'$ and $\angle BAC \cong \angle B'A'C'$, then $\angle DAC \cong \angle D'A'C'$.*

Lemma. *If P and Q are on the same side of $\overleftrightarrow{AB}$, then P is in the interior of $\angle BAQ$ if and only if B and Q are on opposite sides of $\overleftrightarrow{AP}$.*

Lemma. *If P and Q are on the same side of $\overleftrightarrow{AB}$ and $m\angle BAP < m\angle BAQ$, then P is in the interior of $\angle BAQ$.*

Theorem (Supplement postulate) (p. 96, 10/01). *If two angles form a linear pair, then they are supplementary.*

Theorem 5.5 (Vertical angle) (p. 98, 10/01). *If two angles form a vertical pair, then they are congruent.*

Theorem 5.6 (p. 99, 10/01). *If two intersecting lines form one right angle, then they form four right angles.*

Postulate (SAS) (p. 103, 10/08). Given $\triangle ABC$ and $\triangle DEF$. If $\overline{AB} \cong \overline{DE}$, $\angle B \cong \angle E$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.

Theorem 6.2.1 (Isosceles triangle) (p. 104, 10/10). *If two sides of a triangle are congruent, then the angles opposite them are congruent.*

Theorem 6.2.2 (ASA) (p. 106, 10/10). *Given $\triangle ABC$ and $\triangle DEF$. If $\angle A \cong \angle D$, $\overline{AC} \cong \overline{DF}$ and $\angle C \cong \angle F$, then $\triangle ABC \cong \triangle DEF$.*

Theorem 6.2.3 (SSS) (p. 107, 10/10). *Given* $\triangle ABC$ *and* $\triangle DEF$*. If* $\overline{AB} \cong \overline{DE}$*,* $\overline{AC} \cong \overline{DF}$ *and* $\overline{BC} \cong \overline{EF}$*, then* $\triangle ABC \cong \triangle DEF$*.*

Theorem 6.2.4 (p. 109, 10/13). *Every angle has exactly one bisector.*

Theorem 7.7 (SAA) (p. 123, 10/13). *Given* $\triangle ABC$ *and* $\triangle DEF$*. If* $\overline{AB} \cong \overline{DE}$*,* $\angle B \cong \angle E$*, and* $\angle C \cong \angle F$*, then* $\triangle ABC \cong \triangle DEF$*.*

Theorem 7.1 (p. 117, 10/13). *Any exterior angle of a triangle is greater than each of its remote interior angles.*

Theorem 6.5.1/8.3.4 (p. 114/134, 10/13). *Given a line and a point, then there is a unique line which passes through the given point and is perpendicular to the given line.*

Theorem 10.1.1 (p. 148, 10/15). *If two lines are perpendicular to the same line, then the two lines are parallel.*

Theorem 10.1.2 (p. 149, 10/15). *If* P *is a point off line* L*, then there is a line through* P *that is parallel to* L*.*

Theorem 7.2 (p. 119, 10/15). *Given* $\triangle ABC$*. If* $\overline{AB} > \overline{AC}$*, then* $\angle C > \angle B$*.*

Theorem 7.3 (p. 119, 10/15). *Given* $\triangle ABC$*. If* $\angle C > \angle B$*, then* $\overline{AB} > \overline{AC}$*.*

Theorem 7.4 (p. 120, 10/15). *The shortest segment joining a point to a line is the perpendicular segment.*

Theorem 7.5 (Triangular inequality) (p. 120, 10/15). *If* A, B, C *are noncollinear, then* $AB + BC > AC$*.*

Theorem 10.2.1 (p. 151, 10/15). *For any points* A, B, C*,* $AB + BC \geq AC$*.*

Theorem 7.6 (Hinge) (p. 121, 10/20). *Given* $\triangle ABC$ *and* $\triangle DEF$*. If* $\overline{AB} \cong \overline{DE}$*,* $\overline{AC} \cong \overline{DF}$ *and* $\angle A > \angle D$*, then* $\overline{BC} > \overline{EF}$*.*

Theorem 7.8 (Hypotenuse-leg) (p. 124, 10/20). *Given triangles* $\triangle ABC$ *and* $\triangle DEF$ *with* $m\angle A = m\angle D = 90$*. If* $\overline{AB} \cong \overline{DE}$ *and* $\overline{BC} \cong \overline{EF}$*, then* $\triangle ABC \cong \triangle DEF$*.*

Theorem 10.1.3 (p. 150, 10/20). *Given two lines and a transversal. If a pair of alternate interior angles are congruent, then the lines are parallel.*

Theorem 10.1.4 (p. 150, 10/20). *Given two lines and a transversal. If a pair of corresponding angles are congruent, then a pair of alternate interior angles are congruent.*

Theorem 10.1.5 (p. 150, 10/20). *Given two lines and a transversal. If a pair of corresponding angles are congruent, then the lines are parallel.*

Theorem 10.2.2 (Polygonal inequality) (p. 151, 11/22). *If* $A_1, A_2, \ldots, A_n$ *are any points (*$n \geq 2$*), then* $A_1A_2 + A_2A_3 + \cdots + A_{n-1}A_n \geq A_1A_n$*.*

Theorem 10.3.1 (p. 152, 10/22). *The diagonals of a Saccheri quadrilateral are congruent.*

Theorem 10.3.2 (p. 153, 10/22). *Let* $\square ABCD$ *and* $\square A'B'C'D$ *be Saccheri quadrilaterals with lower bases* $\overline{AD}$ *and* $\overline{A'D'}$*. If* $\overline{AD} \cong \overline{A'D'}$ *and* $\overline{AB} \cong \overline{A'B'}$*, then* $\overline{BC} \cong \overline{B'C'}$*,* $\angle B \cong \angle B'$ *and* $\angle C \cong \angle C'$*.*

Theorem 10.3.3 (p. 153, 10/22). *In any Saccheri quadrilateral, the upper base angles are congruent.*

Theorem 10.3.4 (p. 153, 10/27). *In any Saccheri quadrilateral, the upper base is congruent to or longer than the lower base.*

Theorem 10.4.1 (p. 155, 10/27). *In any Saccheri quadrilateral* $\square ABCD$ *with lower base* $\overline{AD}$*, we have* $\angle BDC \geq ABD$*.*

Theorem 10.4.2 (p. 155, 10/27). *If* $\triangle ABD$ *has a right angle at* A*, then* $m\angle B + m\angle D \leq 90$*.*

Theorem 10.4.3 (p. 156, 10/27). *Every right triangle has only one right angle, and its other two angles are acute.*

Theorem 10.4.4 (p. 156, 10/27). *The hypotenuse of a right triangle is longer than either of the legs.*

Theorem 10.4.5 (p. 156, 10/27). *In* $\triangle ABC$*, let* D *be the foot of the perpendicular from* B *to* $\overleftrightarrow{AC}$*. If* $\overline{AC}$ *is the longest side of* $\triangle ABC$*, then* A*-*D*-*C*.*

Theorem 10.4.6 (p. 157, 10/27). *In any triangle* $\triangle ABC$*, we have* $m\angle A + m\angle B + m\angle C \leq 180$*.*

Postulate (Euclidean Parallel Postulate) (p. 160, 10/29). Given a line and an external point, there is only one line which passes through the given point and is parallel to the given line.

Theorem 11.1.1 (p. 160, 10/29). *Given two lines and a transversal. If the lines are parallel, then each pair of alternate interior angles is congruent.*

Theorem 11.1.2 (p. 161, 10/29). *Given two lines and a transversal. If the lines are parallel, then each pair* *of corresponding angles is congruent.*

Theorem 11.1.3 (p. 161, 10/29). *In any triangle $\triangle ABC$ we have $m\angle A + m\angle B + m\angle C = 180$.*

Theorem 11.1.4 (p. 161, 10/29). *The acute angles of a right triangle are complementary.*

Theorem 11.1.5 (p. 161, 10/29). *Every Saccheri quadrilateral is a rectangle.*

Theorem 11.1.6 (p. 162, 10/29). *For any triangle, the measure of an exterior angle is the sum of the measures of its two remote interior angles.*

Theorem 11.1.7 (p. 162, 10/29). *Any two lines parallel to a third line are parallel to each other.*

Theorem 11.1.8 (p. 162, 10/29). *If a transversal is perpendicular to one of two parallel lines, it is perpendicular to the other.*

Theorem 11.1.9 (p. 162, 10/29). *Either diagonal divides a parallelogram into two congruent triangles.*

Theorem 11.1.10 (p. 162, 10/31). *In a parallelogram, each pair of opposite sides are congruent.*

Theorem 11.2.1 (p. 164, 10/31). *Every parallel projection is a one-to-one correspondence.*

Theorem 11.2.2 (p. 164, 10/31). *Parallel projections preserve betweenness.*

Theorem 11.2.3 (p. 165, 10/31). *Parallel projections preserve congruence.*

Theorem 11.4.1 (Basic similarity) (p. 167, 11/03). *Let L_1, L_2, and L_3 be three parallel lines, with common transversals T and T' intersecting them in points A, B, C and A', B', C'. If A-B-C (and A'-B'-C'), then $\frac{BC}{AB} = \frac{B'C'}{A'B'}$.*

Theorem 11.4.2 (p. 170, 11/03). *If two segements on the same line have no points in common, then the ratio of their lengths is preserved under every parallel projection.*

Theorem 11.4.3 (p. 170, 11/03). *Parallel projections preserve ratios.*

Theorem 12.2.1 (AAA similarity) (p. 175, 11/05). *Given $\triangle ABC$ and $\triangle DEF$. If $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$, then $\triangle ABC \sim \triangle DEF$.*

Theorem 12.2.2 (AA similarity) (p. 176, 11/05). *Given a correspondence between two triangles. If two pairs of corresponding angles are congruent, then the correspondence is a similarity.*

Theorem 12.2.3 (SSS similarity) (p. 176, 11/05). *Given two triangles and a correspondence between them. If corresponding sides are proportional, then corresponding angles are congruent, and the correspondence is a similarity.*

Theorem 12.2.4 (SAS similarity) (p. 177, 11/05). *Given a correspondence between two triangles. If two pairs of corresponding sides are proportional, and the included angles are congruent, then the correspondence is a similarity.*

Theorem 12.2.5 (p. 178, 11/05). *Given a similarity between two triangles. If a pair of corresponding sides are congruent, then the correspondence is a congruence.*

Theorem 12.3.1 (p. 178, 11/10). *The altitude to the hypotenuse of a right triangle divides it into two triangles each of which is similar to it.*

Theorem 12.3.2 (Pythagorean) (p. 179, 11/10). *In any right triangle, the square of the length of the hypotenuse is the sum of the squares of the lengths of the other two sides.*

Theorem 12.3.3 (p. 181, 11/10). *Given a triangle whose sides have lengths a, b, and c. If $a^2 + b^2 = c^2$, then the triangle is a right triangle with its right angle opposite the side of length c.*

Theorem 12.3.5 (p. 182, 11/10). *In any triangle, the product of a base and the corresponding altitude is independent of the choice of base.*

Theorem 12.3.6 (p. 183, 11/10). *For similar triangles, the ratio of any two corresponding altitudes is equal to the ratio of any two corresponding sides.*

Remark. The results in sections 24.1 and 24.2 are valid in absolute geometry. That is, they do not depend on either parallel postulate.

Theorem 24.1.1 (p. 371, 11/19). *If $m\angle APD = r_0$, then $\overrightarrow{PD}$ does not intersect $\overrightarrow{AB}$.*

Theorem 24.1.2 (p. 371, 11/19). *If $m\angle APD < r_0$, then $\overrightarrow{PD}$ intersects $\overrightarrow{AB}$.*

Theorem 24.1.3 (p. 372, 11/19). *Let P, A, B and also P', A', B' be as in the definition of the critical number. If $AP = A'P'$, then the critical numbers r_0 and r_0' are the same.*

Theorem 24.1.4 (p. 373, 11/19). *c is a decreasing function. That is, if $a' > a$, then $c(a') \leq c(a)$.*

Theorem 24.1.5 (p. 374, 12/01). *If* $c(a) < 90$*, then* $c(\frac{a}{2}) < 90$.

Theorem 24.1.6 (p. 374, 12/01). *If* $c(a_0) < 90$ *for some* a_0*, then* $c(a) < 90$ *for every* a.

Theorem 24.1.7 (p. 375, 12/01). *If parallels are unique for one line and one external point, then parallels are unique for all lines and all external points.*

Theorem 24.2.4 (p. 378, 12/01). *If* $\overrightarrow{PD}|\overrightarrow{AB}$, $\overrightarrow{PD} \sim \overrightarrow{CD}$ *and* $\overrightarrow{AB} \sim \overrightarrow{QB}$*, then* $\overrightarrow{CD}|\overrightarrow{QB}$.

Theorem 24.2.5 (p. 378, 12/03). *The critical parallel to a given ray, through a given external point, is unique.*

Theorem 24.2.6 (p. 378, 12/03). *If* $\overrightarrow{PD}|\overrightarrow{AB}$*, then* $\triangle DPAB$ *is equivalent to an isosceles open triangle which has* P *as a vertex.*

Theorem 24.2.7 (p. 379, 12/03). *Critical parallelism is a symmetric relation. That is, if* $\overrightarrow{PD}|\overrightarrow{AB}$*, then* $\overrightarrow{AB}|\overrightarrow{PD}$.

Theorem 24.2.8 (p. 380, 12/03). *If* $\overrightarrow{AB}|\overrightarrow{CD}$, $\overrightarrow{CD}|\overrightarrow{EF}$ *and* $\overrightarrow{AB}$ *and* $\overrightarrow{EF}$ *are not equivalent, then* $\overrightarrow{AB}|\overrightarrow{EF}$.

Postulate HPP (Hyperbolic parallel postulate) (p. 139, 12/08). Given a line and an external point, there are at least two lines which passes through the given point and is parallel to the given line.

Theorem 24.3.1 (p. 382, 12/08). *Under HPP, in every closed triangle, each exterior angle is greater than its remote interior angle.*

Theorem 24.3.2 (p. 384, 12/08). *Under HPP, the critical function is strictly decreasing. That is, if* $a' > a$*, then* $c(a') < c(a)$.

Theorem 24.3.3 (p. 384, 12/08). *Under HPP, the upper base angles of a Saccheri quadrilateral are always acute.*

Theorem 24.3.4 (p. 385, 12/08). *Under HPP, in every right triangle* $\triangle ABC$ *we have* $m\angle A + m\angle B + m\angle C < 180$.

Theorem 24.3.5 (p. 385, 12/08). *Under HPP, for every triangle* $\triangle ABC$ *we have* $m\angle A + m\angle B + m\angle C < 180$.

Theorem 24.4.1 (p. 386, 12/08). *Given triangle* $\triangle ABC$ *and* B-D-C*. Then* $\delta\triangle ABC = \delta\triangle ABD + \delta\triangle ADC$.

Theorem 24.4.2 (p. 386, 12/08). *Under HPP, every similarity is a congruence. That is, if* $\triangle ABC \sim \triangle DEF$*, then* $\triangle ABC \cong \triangle DEF$.

Theorem 24.4.3 (p. 387, 12/08). *Under HPP,* $\lim_{a\to\infty} c(a) = 0$.

Theorem 24.4.4 (p. 389, 12/08). *Under HPP, for every number* $\epsilon > 0$ *there is a triangle whose angle sum is less than* ϵ.