Elementary geometry from an advanced point of view

Math 302 — Fall 2008

Suggested solutions — Final exam 12/18

Most answers can be found in the book, so I only record answers for problem 4 here.

Problem 4

a) Show that if D is in the interior of $\triangle ABC$, then $BD + DC < BA + AC$ and $\angle BDC > \angle BAC$.

By crossbar let $\overrightarrow{BD}$ intersext $\overline{AC}$ at E. Then A-E-C and B-D-E. Applying the triangle inequality to $\triangle BAE$ and $\triangle DEC$, we obtain $BA+AE > BE$ and $DE+EC > DC$. Since $AE = AC - EC$ and $DE = BE - BD$, we have $BA + AC > BE + EC$ and $BE + EC > BD + DC$, which implies that $BD + DC < BA + AC$.

Since $\triangle DEC$ has exterior angle $\angle BDC$ with remote interior angle $\angle DEC = \angle BEC$, we have $\angle BDC > \angle BEC$. Similarly, $\angle BEC > \angle BAC$. Thus $\angle BDC > \angle BAC$.

b) Let $\triangle ABC$ be a right triangle, with the right angle at C. Let E be such that B-E-C. Show that there is a point D in the interior of $\triangle ABC$ so that $BD + DE > BA + AC$.

Here the important observation is that D has to be placed very close to A. Since $\angle C$ is a right angle, we know that $AE > AC$. One choice for D could be to let $D \in \overline{AE}$ such that $AD = \frac{1}{2}(AE - AC)$. By the triangle inequality, we know that $BA < BD + AD$. Next $DE = AE - AD$, so that

$$BD + DE > (BA - AD) + (AE - AD) = BA + AE - 2AD = BA + AC.$$