Elementary geometry from an advanced point of view

Math 302 — Fall 2006

Final exam 12/20

This problem set contains ten problems numbered 1a, 1b, 2, 3a, 3b, 3c, 4a, 4b, 5a and 5b. Each problem gives equal credit. Assume absolute geometry, unless noted otherwise.

Problem 1

a) Define congruence between triangles.

b) Prove the isosceles triangle theorem: Given $\triangle ABC$. If $\overline{AB} \cong \overline{AC}$, then $\angle B \cong \angle C$.

Problem 2

Which of the following statements are true, and which are false? (You do <u>not</u> have to give reasons for your answers.)

1. Two different lines may intersect in two points.
2. There exists a triangle whose angle sum is 90 degrees.
3. $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$ and $\overleftrightarrow{CD} \| \overleftrightarrow{EF}$ implies $\overleftrightarrow{AB} \| \overleftrightarrow{EF}$.
4. In euclidean geometry, all similarities are congruences.
5. In hyperbolic geometry, all similarities are congruences.

Problem 3

a) Prove the crossbar theorem: If D is in the interior of $\angle BAC$, then $\overrightarrow{AD}$ intersects $\overline{BC}$, in a point between B and C.

b) Prove the exterior angle theorem: Given $\triangle ABC$. If A-C-D, then $\angle BCD > \angle B$.

c) Given $\triangle ABC$. Let D be the mid-point of $\overline{BC}$, and let E be such that A-D-E and $\overline{AD} \cong \overline{DE}$. Prove that $\triangle ABC$ and $\triangle AEC$ have the same angle sum, and that $m\angle EAC \leq \frac{1}{2} m\angle BAC$ or $m\angle AEC \leq \frac{1}{2} m\angle BAC$.

Problem 4

a) In euclidean geometry, prove the law of cosines: Let $\triangle ABC$ be any triangle, and let D be the foot of the perpendicular from B to $\overleftrightarrow{AC}$. Then

$$(AB)^2 = (BC)^2 + (AC)^2 - 2 \cdot AC \cdot CD.$$

b) In euclidean geometry, prove that the line through the mid-points of two sides of a triangle is parallel to the third side.

Problem 5

a) In hyperbolic geometry, prove that in every right triangle $\triangle ABC$, we have

$$m\angle A + m\angle B + m\angle C < 180.$$

b) In Poincaré's disk model for hyperbolic geometry, draw (sketch) an example of a Saccheri quadrilateral.