

Elementary geometry from an advanced point of view

Math 302 — Fall 2006

Suggested solutions for final exam 12/20

Problem 1

- a) Given $\triangle ABC$ and $\triangle DEF$. The correspondence $ABC \leftrightarrow DEF$ is a congruence if $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, $\overline{BC} \cong \overline{EF}$, $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$.
- b) We have $\overline{AB} \cong \overline{AC}$, $\overline{AC} \cong \overline{AB}$, and $\angle A \cong \angle A$. Hence by SAS $\triangle ABC \cong \triangle ACB$, and in particular $\angle B \cong \angle C$.

Problem 2

1. False.
2. True.
3. False.
4. False.
5. True.

Problem 3

- a) Let F be such that $F-A-C$. Since D is in the interior of $\angle BAC$, B and D are on the same side of \overleftrightarrow{AC} , and by Theorem 4.3.2 \overleftrightarrow{AD} intersects either \overline{FB} or \overline{BC} . The points F and D are on opposite sides of \overleftrightarrow{AB} so by Theorem 4.3.1 \overleftrightarrow{AD} does not intersect \overline{FB} . If \overleftrightarrow{AD} goes through C , then A , D and C have to be collinear, which they are not. Hence \overleftrightarrow{AD} intersects \overline{BC} in a point different from B and C .
- b) Let E be the mid-point of \overline{BC} , and let F be such that $A-E-F$ and $\overline{AE} \cong \overline{EF}$. By SAS $\triangle AEB \cong \triangle FEC$, and $\angle B \cong \angle BCF$. By Theorem 4.2.7 F is in the interior of $\angle BCD$, so $\angle B \cong \angle BCF < \angle BCD$.

- c) Let α denote the angle sum of a triangle. By SAS $\triangle ABD \cong \triangle ECD$, so $\alpha\triangle ABD = \alpha\triangle ECD$. Furthermore,

$$\alpha\triangle ABC = \alpha\triangle ABD + \alpha\triangle ACD - 180 = \alpha\triangle ECD + \alpha\triangle ACD - 180 = \alpha\triangle AEC.$$

Because $\triangle ABD \cong \triangle ECD$, we have $\angle BAE \cong \angle AEC$. Therefore $m\angle BAC = m\angle BAE + m\angle EAC = m\angle AEC + m\angle EAC$. So if $m\angle EAC > \frac{1}{2}m\angle BAC$, it follows that $m\angle AEC = m\angle BAC - m\angle EAC < \frac{1}{2}m\angle BAC$.

Problem 4

- a) We need to consider two different cases. If $D = A$ or $A-D-C$, then $AC = AD + CD$. By Pythagoras $(AB)^2 = (AD)^2 + (BD)^2$ and $(BC)^2 = (BD)^2 + (CD)^2$. Hence

$$\begin{aligned} (BC)^2 + (AC)^2 - 2 \cdot AC \cdot CD \\ &= (BD)^2 + (CD)^2 + (AD)^2 + 2 \cdot AD \cdot CD + (CD)^2 - 2 \cdot (AD + CD) \cdot CD \\ &= (BD)^2 + (AD)^2 = (AB)^2. \end{aligned}$$

If $D-A-C$, then $AC = CD - AD$, and similar calculations as above can be done.

- b) Given $\triangle ABC$, let D be the mid-point of \overline{AB} and let E be the mid-point of \overline{AC} . By the SAS similarity theorem $\triangle ABC \sim \triangle ADE$, so $\angle ADE \cong \angle ABC$. It then follows from Theorem 10.1.5 that $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$.

Problem 5

- a) Seeking a contradiction, we assume that $m\angle A + m\angle B + m\angle C = 180$ (here we have used Theorem 10.4.6). Let $\angle A$ be the right angle, so that $\angle B$ and $\angle C$ are complementary. Let D be the point on the opposite side of \overleftrightarrow{BC} from A such that $\angle BCD \cong \angle ABC$ and $\overline{CD} \cong \overline{AB}$. Then $\square ABCD$ is a Saccheri quadrilateral. By SAS $\triangle ABC \cong \triangle DCB$, so $\angle D$ is a right angle, which contradicts Theorem 24.3.3.

- b)