Differential equations

Math 217 — Fall 2009

November 17

This exam contains thirteen problems numbered 1 through 13. Problems 1 – 12 are multiple choice problems. Problem 13 is a free-response question.

Problem 1

Suppose x gives the position of an undamped mass-spring system where $m = 2$ and $k = 32$. An external force of $F_E(t) = 10\sin(\omega t)$ is applied to the system. For what value of ω will resonance occur?

A) $\omega = 2$ **B)** $\omega = 2\sqrt{2}$ **C)** $\omega = 4$ **D)** $\omega = 4\sqrt{2}$ **E)** $\omega = 8$

F) $\omega = 16$

Problem 2

Box 1 with mass 1 is vertically attached to the ceiling of a room via a spring with the spring constant being k_1. Box 2 with mass 2 is attached to Box 1 via a spring with the spring constant being k_2. Let $x_1(t)$ be the displacement of Box 1 from the static equilibrium position and $x_2(t)$ be the displacement of Box 2 from the static position. Find the system of differential equations that $x_1(t)$ and $x_2(t)$ satisfy.

A) $x_1'' = k_1x_1 + k_1(x_2 - x_1), 2x_2'' = k_2(x_2 - x_1)$

B) $x_1'' = -k_1x_1 + k_2(x_2 + x_1), 2x_2'' = k_2(x_2 - x_1)$

C) $x_1'' = -k_1x_1 + k_2(x_2 - x_1), 2x_2'' = -k_2(x_2 - x_1)$

D) $x_1'' = -k_1x_1 + k_2(x_2 + x_1), 2x_2'' = -k_2(x_2 - x_2)$

E) $x_1'' = -k_2x_2 + k_1(x_2 - x_1), 2x_2'' = -k_1(x_2 - x_1)$

F) $x_1'' = -k_2x_2 + k_1(x_2 + x_1), 2x_2'' = -k_1(x_2 - x_1)$

Problem 3

Write the following differential equation as a system of first order differential equations

$$x^{(3)} + x'' + \sin(t)x' + x^2 + t^2 = e^{t^2}$$

A) $x_1' = x_2,\ x_2' = x_3 + \sin(t)x_2,\ x_3' = e^{t^2} - (x_1^2 + t^2)$

B) $x_1' = x_1,\ x_2' = x_2,\ x_3' = e^{t^2} - \sin(t)x_2 - (x_1^2 + t^2)$

C) $x_1' = x_1^2 + t^2,\ x_2' = \sin(t)x_2,\ x_3' = e^{t^2}$

D) $x_1' = x_1^2 + t^2,\ x_2' = x_3 + \sin(t)x_1,\ x_3' = x_3$

E) $x_1' = x_1,\ x_2' = x_2,\ x_3' = e^{t^2} - (x_1^2 + t^2) - \sin(t)x_2$

F) $x_1' = x_2,\ x_2' = x_3,\ x_3' = e^{t^2} - (x_1^2 + t^2) - \sin(t)x_2 - x_3$

Problem 4

Find a general solution of the system

$$\begin{aligned} \frac{dx_1}{dt} &= 4x_1 - 3x_2 \\ \frac{dx_2}{dt} &= 3x_1 + 4x_2. \end{aligned}$$

A) $e^{4t}\begin{bmatrix} c_1\cos(3t) + c_2\sin(3t) \\ c_1\sin(3t) + c_2\cos(3t) \end{bmatrix}$

B) $e^{4t}\begin{bmatrix} c_1\cos(3t) - c_2\sin(3t) \\ c_1\sin(3t) + c_2\cos(3t) \end{bmatrix}$

C) $e^{4t}\begin{bmatrix} c_1\cos(3t) + c_2\cos(3t) \\ c_1\sin(3t) + c_2\cos(3t) \end{bmatrix}$

D) $e^{4t}\begin{bmatrix} c_1\cos(3t) - c_2\cos(3t) \\ c_1\sin(3t) + c_2\cos(3t) \end{bmatrix}$

E) $e^{4t}\begin{bmatrix} c_1\sin(3t) + c_2\sin(3t) \\ c_1\sin(3t) + c_2\cos(3t) \end{bmatrix}$

F) $e^{4t}\begin{bmatrix} c_1\sin(3t) - c_2\sin(3t) \\ c_1\sin(3t) + c_2\cos(3t) \end{bmatrix}$

Problem 5

Suppose the 3×3 matrix $\mathbf{A}$ has an eigenvalues of $\lambda = 1, 2$ and

$$\mathbf{A}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Which of the following are true?

I) $[1\ 0\ 1]^T$ is an eigenvector associated with $\lambda = 1$.

II) $\mathbf{x} = [1\ 1\ 0]^T e^t$ is a solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

III) $\mathbf{x} = [0\ -1\ 1]^T te^t$ is a solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

IV) $\mathbf{x} = ([1\ 1\ 0]^T + [1\ 1\ 0]^T t)e^t$ is a solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$

A) I and II **B)** II and III **C)** III and IV **D)** I, II and III

E) I, II and IV **F)** II, III, and IV

Problem 6

Find the solution to the initial value problem

$$\mathbf{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

A) $\frac{1}{7}\begin{bmatrix} 3e^{-2t} + 4e^{5t} \\ -9e^{-2t} + 2e^{5t} \end{bmatrix}$ **B)** $\frac{1}{7}\begin{bmatrix} 4e^{-2t} + 3e^{5t} \\ 2e^{-2t} - 9e^{5t} \end{bmatrix}$ **C)** $\frac{1}{7}\begin{bmatrix} 9e^{-2t} - 2e^{5t} \\ -4e^{-2t} - 3e^{5t} \end{bmatrix}$

D) $\frac{1}{7}\begin{bmatrix} 3e^{2t} + 4e^{-5t} \\ -9e^{2t} + 2e^{-5t} \end{bmatrix}$ **E)** $\frac{1}{7}\begin{bmatrix} 4e^{2t} + 3e^{-5t} \\ 2e^{2t} - 9e^{-5t} \end{bmatrix}$ **F)** $\frac{1}{7}\begin{bmatrix} 9e^{2t} - 2e^{-5t} \\ -4e^{2t} - 3e^{-5t} \end{bmatrix}$

Problem 7

Find a solution of the following equation

$$\mathbf{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \mathbf{x},$$

satisfying the initial values $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

A) $\begin{bmatrix} 3 \\ 0 \end{bmatrix} e^{4t}$ **B)** $\begin{bmatrix} 3 \\ 0 \end{bmatrix} e^{-4t}$ **C)** $\begin{bmatrix} 3t+1 \\ 3t \end{bmatrix} e^{4t}$

D) $\begin{bmatrix} (1-3t) \\ 3t \end{bmatrix} e^{-4t}$ **E)** $\begin{bmatrix} (3-t) \\ 3t \end{bmatrix} e^{4t}$ **F)** $\begin{bmatrix} (1-3t) \\ 3t \end{bmatrix} e^{4t}$

Problem 8

Find the transient motion and steady periodic oscillations of a damped mass-and-spring system with $m = 1, c = 2$, and $k = 26$ under the influence of an external force $F(t) = 82\cos 4t$ with $x(0) = 6$ and $x'(0) = 0$. What is the amplitude of the steady periodic oscillation of the mass?

A) $\sqrt{38}$ **B)** $\sqrt{39}$ **C)** $\sqrt{40}$ **D)** $\sqrt{41}$ **E)** $\sqrt{42}$ **F)** $\sqrt{43}$

Problem 9

What is the form of a particular solution to the nonhomogeneous system of equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2e^{-t} \\ te^{2t} \end{bmatrix}$$

A) $\mathbf{x}_p = \mathbf{a}e^{-t} + \mathbf{b}te^{2t}$

B) $\mathbf{x}_p = \mathbf{a}te^{-t} + (\mathbf{b} + \mathbf{c}t)e^{2t}$

C) $\mathbf{x}_p = \mathbf{a}e^{-t} + (\mathbf{b} + \mathbf{c}t)e^{2t}$

D) $\mathbf{x}_p = \mathbf{a}te^{-t} + (\mathbf{b}t + \mathbf{c}t^2)e^{2t}$

E) $\mathbf{x}_p = (\mathbf{a} + \mathbf{b}t)e^{-t} + (\mathbf{c}t + \mathbf{d}t^2)e^{2t}$

F) $\mathbf{x}_p = (\mathbf{a} + \mathbf{b}t)e^{-t} + (\mathbf{c} + \mathbf{d}t + \mathbf{e}t^2)e^{2t}$

Problem 10

Find a particular solution to the nonhomogeneous system

$$\mathbf{x}' = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 4 \\ e^{2t} \end{bmatrix}.$$

A) $\begin{bmatrix} -2 + t^2e^{2t} \\ e^{2t} \end{bmatrix}$ **B)** $\begin{bmatrix} t^2e^{2t} \\ e^{2t} \end{bmatrix}$ **C)** $\begin{bmatrix} 2 + t^2e^{2t} \\ e^{2t} \end{bmatrix}$ **D)** $\begin{bmatrix} -2 + t^2e^{2t} \\ te^{2t} \end{bmatrix}$

E) $\begin{bmatrix} t^2e^{2t} \\ te^{2t} \end{bmatrix}$ **F)** $\begin{bmatrix} 2 + t^2e^{2t} \\ te^{2t} \end{bmatrix}$

Problem 11

A complementary solution of

$$y'' + y = \frac{1}{\cos x}$$

is $y_C = C_1 \cos x + C_2 \sin x$. Find a particular solution.

Hint: The formula

$$\int \frac{\sin x}{\cos x}\, \mathrm{d}x = -\ln|\cos x| + C$$

may be useful.

A) $\frac{1}{\cos x}$ **B)** $x \sin x$ **C)** $\cos x \cdot \ln|\cos x|$ **D)** $x \sin x + \cos x \cdot \ln|\cos x|$
E) $x \sin x - \cos x \cdot \ln|\cos x|$ **F)** $\cos x + x \sin x$

Problem 12

Suppose **B** is a 6×6 matrix with a characteristic equation of

$$|\mathbf{B} - \lambda\mathbf{I}| = (\lambda - 1)^3(\lambda + 1)^2(\lambda - 2) = 0,$$

and that the eigenvalue $\lambda = 1$ is defective. Which of the following is/are *not* true?

I) The matrix **B** has 3 linearly independent eigenvectors.

II) The eigenvalue $\lambda = 1$ has 3 linearly independent eigenvectors.

III) There are 6 linearly independent solutions to the differential equation $\mathbf{x}' = \mathbf{B}\mathbf{x}$.

A) I only **B)** II only **C)** III only **D)** I and II **E)** I and III

F) II and III

The following problem is a free-response question. You should justify your answers.

Problem 13

a) Find the general solution of

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}$$

given that $|\mathbf{A} - \lambda\mathbf{I}| = -(\lambda - 2)^3$.

b) Solve the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$