Differential equations

Math 217 — Fall 2009

Exam September

This practice exam contains fourteen problems numbered 1 through 14. Problems 1 – 13 are multiple choice problems. Problem 14 is a free-response question.

Problem 1

Suppose $f(x, y)$ and $f_y(x, y)$ are continuous functions for all x and y, which of the following is a true statement?

A) The initial value problem $y' = f(x, y), y(1) = 1$ does not have solution.

B) The initial value problem $y' = f(x, y), y(1) = 1$ has a unique solution for every x.

C) The differential equation $y' = f(x, y)$ is separable.

D) The initial value problem $y' = f(x, y), y(1) = 1$ has a unique solution in an interval I containing 1.

E) There exist two functions y_1 and y_2 such that $\frac{dy_1}{dx} = f(x, y_1(x))$ and $\frac{dy_2}{dx} = f(x, y_2(x))$ and $y_1(1) = 1 = y_2(1)$, but $y_1(x) \neq y_2(x)$ for $x \neq 1$.

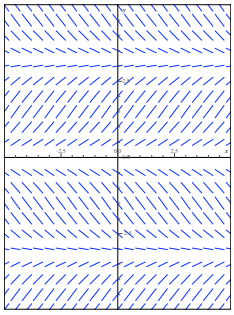
F) None of the above.

Problem 2

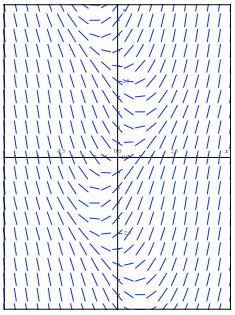
Identify the slope field of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -x^2 + \sin y.$$

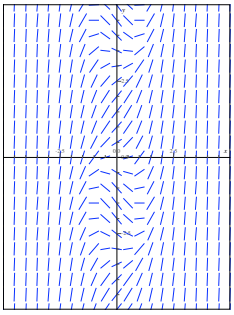
A)



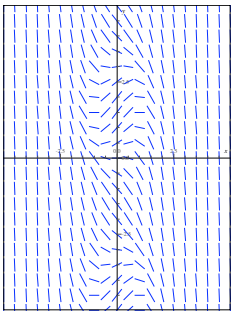
B)



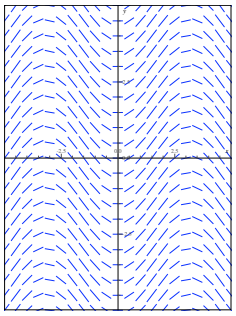
C)



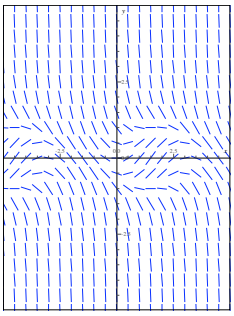
D)



E)



F)



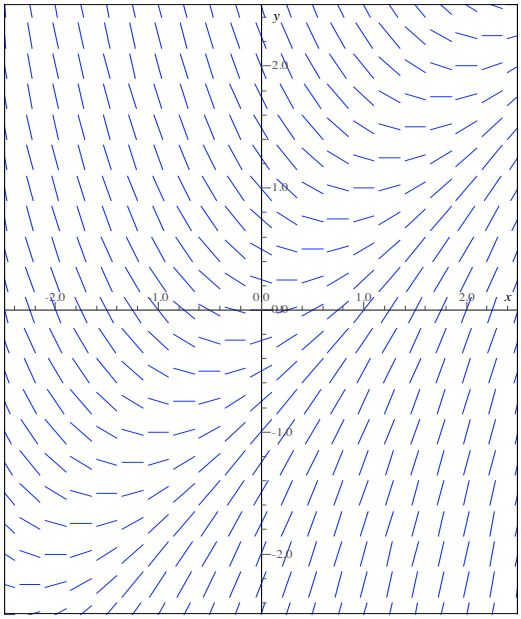
Problem 3

Suppose a car traveling at 60 mi/h(= 88 ft/s) slams on the brakes applying a constant deceleration of 22 ft/s^2. How far will the car skid?

A) 352 ft **B)** 4 ft **C)** 176 ft **D)** 88 ft **E)** 528 ft **F)** None of the above

Problem 4

Which of the following statements about the differential equation $\frac{dy}{dx} = f(x, y)$ with slope field as below are true?



I) $y(x) = x + 1$ is a solution.

II) The solution passing through $(2, 1)$ is increasing.

III) For the solution passing through $(0, -1)$, $y(1) \approx 0$.

A) Only I **B)** Only II **C)** Only III **D)** I and II **E)** I and III **F)** II and III

Problem 5

If y is defined implicitly by the equation $xy^2 + x^3 - y = 2$, then which of the following differential equations does y satisfy?

A) $y' = y^2 + 3x$ **B)** $(2xy - 1)y' = -3x^2$ **C)** $x + yy' = 0$ **D)** $(2xy - 1)y' = -3x^2 - y^2$ **E)** $y' = 2xy + 3x^2$ **F)** None of the above

Problem 6

Find a solution of the initial value problem

$$xy' = y + \sqrt{x^2 - y^2}, \qquad y(1) = 0$$

for $x > 0$.

Hint: One or several of the following integrals may be useful,

$$\int \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \arcsin u + C, \qquad \int \frac{\mathrm{d}u}{1-u^2} = \tfrac{1}{2}\ln\left|\tfrac{u+1}{u-1}\right| + C.$$

A) $y = x\sin(\ln x)$ **B)** $y = x\cos(\ln x)$ **C)** $y = \sin(\ln x)$ **D)** $y = \cos(\ln x)$
E) $y = x^2\sin(\ln x)$ **F)** $y = x^2\cos(\ln x)$

Problem 7

Suppose a water tank initially contains a 100 gal mixture of salt and water. Brine at a concentrate of .5 lbs/gal flows into the tank at the same rate the well mixed brine flows out of the tank. How much salt should initially be in the tank so that the amount of salt stays constant as the brine flows through the tank i.e. so equilibrium is acheived?

A) 100 lbs **B)** 200 lbs **C)** 150 lbs **D)** 50 lbs **E)** 0 lbs **F)** None of the above

Problem 8

Which of the following differential equations is separable?

A) $y' - e^y = e^{x+y}$ **B)** $x\frac{dy}{dx} - 2y = x + 2y$ **C)** $xy\frac{dy}{dx} = x^3 + y^3$ **D)** $(x+y)y' = x - y$ **E)** $y' = \ln(xy)$ **F)** None of the above

Problem 9

Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = \frac{\cos x}{e^x} \qquad y(0) = 1.$$

What is $y(\pi)$?

A) $2e^\pi$ **B)** $e^\pi + 1$ **C)** e^π **D)** $\frac{2}{e^\pi}$ **E)** $\frac{1}{e^\pi} + 1$ **F)** $\frac{1}{e^\pi}$

Problem 10

One of the following equations are exact. Identify which one, and find a solution $y(x)$ of that equation satisfying $y(0) = 0$. What is $y(1)$?

$$\big(2x + 2y\big) + \big(2x + 2y\big)\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \qquad \big(2x + 3y\big) + \big(2x + 3y\big)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

A) -2 **B)** -1 **C)** 0 **D)** 1 **E)** 2 **F)** 3

Problem 11

If $y' + x = xy$ and $y(0) = 1$ what is $y(2)$?

A) 0 **B)** -1 **C)** 2 **D)** 1 **E)** -2 **F)** None of the above

Problem 12

At any time t, the rate of growth of the population N of elk in Lone Elk Park is proportional to the product of N and $L - N$, where $L = 500$ is the maximum number of elk the park can maintain. At $t = 0$ there are 100 elk in the park, and at $t = 4$ there are 200 elk in the park. Find the number of elk when $t = 1$?

A) 100 **B)** 120 **C)** 121 **D)** 150 **E)** 200 **F)** 500

Problem 13

Suppose that a river is $2a = 1$ mile wide and that its midstream velocity is $v_0 = 9mi/h$. If a swimmer's velocity is $v_s = 3mi/h$. Assume that the trajectory of the swimmer satisfies the differential equation

$$\frac{dy}{dx} = \frac{v_0}{v_s}(1 - \frac{x^2}{a^2}).$$

How far does the swimmer drift when he crosses the river?

A) 0.5 mile **B)** 1 mile **C)** 1.5 miles **D)** 2miles **E)** 2.5 miles **F)** 3 miles

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The following problem is a free-response question. You should justify your answers.

Problem 14

Consider a cascade of two tanks with $V_1 = 100$(gal) and $V_2 = 200$(gal) the volumes of brine in the two tanks. Each tank also initially contains 50 lb of salt. Pure water flows into tank 1 at the rate of 5 gal/min and perfect mixed solution enters from tank 1 to tank 2 and leaves from tank 2 at the same rate of 5 gal/min.

a) Find the amount $x(t)$ of salt in tank 1 at time t.

b) Suppose that $y(t)$ is the amount of salt in tank 2 at time t.

c) Find the maximum amount of salt in tank 2.