

**Math 331 Spring 2006**  
**Midterm 2 (By April.17th 2:00)**

In each number choose either (a) or (b) and show your work in detail.  
In each question, (a) and (b) are 20 points and 25 points respectively.

1. (a) Consider  $5\mathbb{Z}$  and  $30\mathbb{Z}$  in  $\mathbb{Z}$ . Show that

$$5\mathbb{Z}/30\mathbb{Z} \cong \mathbb{Z}_6.$$

- (b) For any positive integers  $k$  and  $m$ , show that

$$k\mathbb{Z}/(km)\mathbb{Z} \cong \mathbb{Z}_m.$$

2. (a) For  $A_n$  with  $n \geq 5$ , prove or disprove that there is no subgroup of  $A_n$  with index 2.  
(b) Let  $P$  be a  $p$ -group. Let  $A$  be a normal subgroup of order  $p$ . Prove that  $A$  is contained in the center of  $P$ .
3. (a) Show that every group of order 15 is cyclic.  
(b) Let  $G$  be a group with order  $mp$ , where  $p$  is prime and  $1 < m < p$ . Prove that  $G$  is not simple.
4. (a) How many conjugacy classes of 5-cycles in  $S_5$  ?  
(b) How many conjugacy classes of 2-cycles in  $S_5$  ?
5. (a) Show that the set of all nilpotent elements in a commutative ring  $R$  is an ideal. ( $x$  in  $R$  is called nilpotent if  $x^n = 0$  for some  $n \in \mathbb{N}$ ).  
(b) Let  $I$  be an ideal in a commutative ring  $R$  and let

$$\text{Rad}(I) := \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}\}.$$

Show that  $\text{Rad}(I)$  is an ideal of  $R$ .