Math 429 Fall 2005
Midterm 1 (By Oct.10th 12:00)

1. (15 points) Let A be an invertible element in $Mat_{n\times n}(\mathbb{R})$ and $a_1, a_2, .. \, a_k$ be linearly independent vectors in $\mathbb{R}^n$ with $k \leq n$. Show $Aa_1, Aa_2, .. \, Aa_k$ are linearly independent.

2. (30 points) Let S be the set of symmetric matrices defined as

$$S = \left\{ A \in Mat_{n\times n}(\mathbb{R}) \mid A^T = A \right\}.$$

   Is this a vector subspace of $Mat_{n\times n}(\mathbb{R})$? If so, what is the dimension of S.

3. (35 points) Consider a linear transformation A: $\mathbb{R}^{2005} \to \mathbb{R}^{2005}$ defined as

$$A = \begin{pmatrix} 1 & 2 & 3 & ... & 2005 \\ 2 & 2^2 & 2\times 3 & ... & 2\times 2005 \\ 3 & 3\times 2 & 3^2 & ... & 3\times 2005 \\ . & . & . & ... & . \\ 2005 & 2005\times 2 & 2005\times 3 & ... & 2005^2 \end{pmatrix}.$$

   Find $\operatorname{Im}(A)$, $rk(A)$ and $n(A)$. Hint, observe that A can be written as

$$\begin{pmatrix} 1 & 2 & 3 & ... & 2005 \end{pmatrix}^T \begin{pmatrix} 1 & 2 & 3 & ... & 2005 \end{pmatrix}.$$

   * Choose one of the following two questions and show your work on it. In the following two questions, let $GL(n, \mathbb{R})$ be the set of invertible matrices in $Mat_{n\times n}(\mathbb{R})$ and you may use the fact that a $GL(n, \mathbb{R})$ is a group with matrix multiplication without proof.

4. (20 points) Let $\tilde{S}$ be defined as

$$\tilde{S} = \left\{ A \in GL(n, \mathbb{R}) \mid A^T = A \right\}.$$

   Prove or disprove that $\tilde{S}$ is a group with matrix multiplication.

5. (30 points) Let T be a nonzero element in $Mat_{n\times n}(\mathbb{R})$. $\mathcal{P}$ is defined as

$$\mathcal{P} = \{ P \in GL(n, \mathbb{R}) \mid P(\ker T) = \ker T, P(\operatorname{Im} T) = \operatorname{Im} T \}.$$

   Prove or disprove that $\mathcal{P}$ is a group with matrix multiplication.

   Note, $P(\ker T)$ and $P(\operatorname{Im} T)$ are defined as

$$P(\ker T) = \{ Pa \mid a \in \ker T \} \subset \mathbb{R}^n$$
$$P(\operatorname{Im} T) = \{ Pb \mid b \in \operatorname{Im} T \} \subset \mathbb{R}^n.$$

   Hint. At first, you need to see $P^{-1} \in \mathcal{P}$ if $P \in \mathcal{P}$.