

**Math 331 Spring 2006**  
**Final (By May.3rd 2:00)**

\* Work all the questions between 1 and 6. Show your work in detail.

1. (15 points) Let  $K$  be a subgroup of a group  $G$ . Show  $N_G(K)$  defined as

$$N_G(K) := \{g \in G \mid gKg^{-1} \subset K\}$$

is a subgroup of  $G$ .

2. (15 points) Find all the groups of order 9 up to group isomorphism.  
3. (20 points) Let  $G$  be a group with order 20. Prove that  $G$  is not simple.  
4. (20 points) Show that the center of  $S_n$  is the identity subgroup for  $n > 2$ .  
5. (20 points) Show that

$$\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}$$

is a rational number.

6. (20 points) Is  $109\mathbb{Z}$  in  $\mathbb{Z}$  a maximal ideal? Justify your answer.

\* Choose three questions from the following questions and show your work in detail on those.

7. (30 points) Let  $G$  be a finite group and  $H$  be a subgroup of  $G$  of order  $n$ . If  $H$  is the only subgroup of  $G$  of order  $n$ , then  $H$  is normal in  $G$ .  
8. (35 points) A group  $G$  is abelian if and only if the map  $G \rightarrow G$  given by  $x \mapsto x^{-1}$  is an automorphism.  
9. (30 points) Find all the group  $G$  of order 14 up to isomorphism.  
10. (30 points) Let  $P$  be a  $p$ -group. Let  $A$  be a normal subgroup of order  $p$ . Prove that  $A$  is contained in the center of  $P$ . (Hint: observe all the subgroup of order  $p$  in  $P$  are conjugate.)  
11. (40 points) Let  $x$ ,  $y$  and  $z$  be rational numbers such that

$$x + y\sqrt[3]{2} + z\sqrt[3]{4} = 0.$$

Prove  $x = y = z = 0$ .