Math 429 Fall 2005

Final Exam (By Dec.14th 12:00)

* Your final exam must be handed in by yourself and **No late exam will be accepted***

Solve or prove either all even or all odd numbers between question 1 and 10.(Each 20 points)

1. Let A be a 2×2 real matrix defined as
$$\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$
Consider $A^{2005} = aA + bI_2$. Determine a and b.

2. Let A be a 2×2 real matrix defined as
$$\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$
Determine if A is diagonalizable.

3. Let $f(x)$ and $g(x)$ be polynomials in $F[x]$ where F is $\mathbb{R}$ defined as
$$f(x) = x^3 - 1$$
$$g(x) = x^5 - x^4 + x^3 - x^2 + x - 1$$
Find the *g.c.d.* of $f(x)$ and $g(x)$.

4. Is it true that there is an $n \times n$ matrix whose minimal polynomial has degree 0. Justify your answer.

5. Let A be an $n \times n$ matrix whose minimal polynomial has degree 1. Describe all the possible A.

6. Find the minimal polynomial of N_{2005} where N_r is an $r \times r$ matrix defined as
$$\begin{pmatrix} 0 & 0 & .. & 0 & 0 \\ 1 & 0 & .. & 0 & 0 \\ 0 & 1 & .. & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & .. & 1 & 0 \end{pmatrix}.$$

7. Let A be an invertible skew-symmetric matrix. Prove or disprove that A^{-1} is also a skew-symmetric matrix.

8. Let A be a skew-symmetric matrix, show that A^2 is a symmetric matrix.

9. Find the Jordan form of
$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$

10. Let A be a 3×3 matrix with unique eigenvalue a. When eigenspace of a has only 1 dimension, describe the Jordan form of A.

* Solve or prove **ONLY** three questions from the following questions.

11. (30 points) Prove or disprove that if A is a real skew-symmetric $n \times n$ matrix, then $I_n + A$ is an invertible matrix. (Hint, for a symmetric real matrix, its eigenvalues are real number.)

12. (35 points) Prove or disprove that for a square matrix A, A and A^T are similar matrices.

13. (35 points) For any real matrix A, prove or disprove that $rk\left(A^T A\right) = rk\,(A)$.

14. (35 points) Find the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots & \dots \\ a_2 a_3 \dots a_n & a_1 a_3 \dots a_n & a_1 a_2 \dots a_n & & a_1 a_2 \dots a_{n-1} \end{pmatrix}.$$

15. (30 points) Let A be a $n \times n$ matrix with $rk\,(A) = 1$. Show that $\det(A + I_n) = tr\,(A) + 1$.

16. (30 points) Show that any 2×2 unitary matrix with determinant 1 is of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

, where α and β are complex numbers with $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$.

17. (35 points) Consider the following vectors in $M_{2\times 2}(\mathbb{R})$ equipped with the standard inner product. Find an orthonormal basis of the subspace spanned by these vectors.

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \text{ and } \begin{pmatrix} 2005 & 2005 \\ 2005 & -2003 \end{pmatrix}.$$

18. (50 points) Let A be a $n \times n$ matrix. A is nilpotent if and only if $tr\,(A^r) = 0$ for $r = 1, ..., n$.

***If you believe your homework and exams don't show your work properly, you may have an oral test. Oral test will be taken in my office Dec. 14th, 10:30 -12:00 and 1:00 - 2:00. If you want to take, please let me know beforehand. *Warning* ! This is an option and you may have minus points.($-30 \leq$ points≤ 30)**