Math 429 Fall 2005
Assignment 4: Due by Oct 3rd

1. Consider a linear transformation T: $\mathbb{R}^5 \to \mathbb{R}^4$ defined as

$$\begin{aligned} y_1 &= x_1 + x_2 + x_3 + x_4 + x_5, \\ y_2 &= 3x_1 + 2x_2 + x_3 + x_4 - 3x_5, \\ y_3 &= x_2 + 2x_3 + 2x_4 + 6x_5, \\ y_4 &= 5x_1 + 4x_2 + 3x_3 + 3x_4 - x_5 \end{aligned}$$

   where $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ and $(y_1, y_2, y_3, y_4) \in \mathbb{R}^4$. Describe $\ker(T)$ and $\mathrm{Im}\,(T)$. Find another linear transformation $S \in Hom\,(\mathbb{R}^5, \mathbb{R}^4)$ such that $\ker(S) = \ker(T)$ and $\mathrm{Im}\,(S) = \mathrm{Im}\,(T)$.

2. Let T be an element in $Hom\,(\mathbb{R}^n, \mathbb{R}^n)$ and P be an invertible $n \times n$ real matrix. Consider $S = PTP^{-1}$.

   (1) Prove or disprove

$$\begin{aligned} \ker S &= \ker TP^{-1} \\ \mathrm{Im}\, S &= \mathrm{Im}\, PT. \end{aligned}$$

   (2) $\mathcal{P}$ is defined as

$$\mathcal{P} = \left\{ P \in Mat_{n\times n}(\mathbb{R}) \mid \ker PTP^{-1} = \ker T, \mathrm{Im}\, PTP^{-1} = \mathrm{Im}\, T \right\}.$$

   Prove or disprove that $\mathcal{P}$ is a group and describe elements in $\mathcal{P}$. Note, you may use the fact that a set of invertible matrices is a group without proof.