Homework 10

Problem 1

Problem 2(# 5.34)

i

For F and G in $\mathscr{F}_0$

$$T(G) - T(F) = \int xJ(G(x))dG(x) - \int xJ(F(x))dF(x)$$

$$= \int_0^1 [G^{-1}(t) - F^{-1}(t)]J(t)dt$$

$$= \int_0^1 \int_{F^{-1}(t)}^{G^{-1}(t)} dxdJ(t)dt = \int_{-\infty}^{\infty} \int_{G(x)}^{F(x)} J(t)dtdx$$

$$= \int_{-\infty}^{\infty} [F(x) - G(x)]J(F(x))dx - \int_{-\infty}^{\infty} U_G(x)[G(x) - F(x)]J(F(x))dx,$$

where

$$U_G(x) = \begin{cases} \dfrac{\int_{F(x)}^{G(x)} J(t)dt}{[G(x) - F(x)]J(F(x))} - 1, G(x) \neq F(x), J(F(x)) \neq 0 \\ 0 \end{cases}$$

and the fourth equality follows from Fubini's theorem and the fact that the region in R^2 between curves $F(x)$ and $G(x)$ is the same as the region in R^2 between curves $G^{-1}(t)$ and $F^{-1}(t)$. Then for any $\Delta \in \mathscr{D} = \{c(G_1 - G_2) : c \in \mathscr{R}, G_j \in \mathscr{F}, j = 1, 2\}$,

$$\lim_{t \to 0} \frac{T(F + t\Delta) - T(F)}{t} = - \int_{-\infty}^{\infty} \Delta(x)J(F(x))dx,$$

since $\lim_{t \to 0} U_{F+t\Delta}(x) = 0$ and, by the dominated convergence theorem,

$$\lim_{t \to 0} \int_{-\infty}^{\infty} U_{F+t\Delta}\Delta(x)J(F(x))dx = 0.$$

Letting $\Delta = \delta_x - F$, we obtain the influence function as claimed.

ii

By Fubini's theorem,

$$\int \phi_F(x)dF(x) = -\int_{-\infty}^{\infty}[\int(\delta_x - F)(y)dF(x)]J(F(y))dy = 0,$$

since $\int \delta_x(y)dF(x) = F(y)$. Suppose now that $|J| < C$ for a constant C. Then

$$\begin{aligned}|\phi_F(x)| &\le C\int_{-\infty}^{\infty}|\delta_x(y) - F(y)|dy \\ &= C(\int_{-\infty}^{x}F(y)dy + \int_{x}^{\infty}[1 - F(y)]dy) \\ &\le C(|x| + \int_{-\infty}^{0}F(y)dy + \int_{0}^{\infty}[1 - F(y)]dy) \\ &= C(|x| + E|X|),\end{aligned}$$

where X is the random variable having distribution F. Thus,

$$[\phi_F(x)]^2 \le C^2(|x| + E|X|)^2$$

and $\int[\phi_F(x)]^2 dF(x) < \infty$ when $EX^2 < \infty$.

(# 5.37) i

When $J \equiv 1$, $T(G) = \int xdG(x)$ is the mean functional ($\mathscr{F}_0$ is the collection of all distribution with finite means). From the previous exercise, the influence function is

$$\phi_F(x) = -\int_{-\infty}^{\infty}[\delta_x(y) - F(y)]dy = \int_{-\infty}^{x}F(y)dy - \int_{x}^{\infty}[1 - F(y)]dy$$

$$= \int_{0}^{x}dy + \int_{\infty}^{0}F(y)dy - \int_{0}^{\infty}[1 - F(y)]dy = x - \int_{-\infty}^{\infty}ydF(y)$$

This influence function is continuous, but not bounded.

ii

When $J(t) = 4t - 2$,

$$\phi_F(x) = 2\int_{-\infty}^{x}F(y)[2F(y) - 1]dy - 2\int_{x}^{\infty}[1 - F(y)][2F(y) - 1]dy$$

Clearly, ϕ_F is continuous, Since

$$lim_{x\to\infty}\int_{-\infty}^{x}F(y)[2F(y) - 1]dy = \int_{-\infty}^{\infty}F(y)[2F(y) - 1]dy = \infty$$

and

$$lim_{x\to\infty}\int_{x}^{\infty}[1 - F(y)][2F(y) - 1]dy = 0,$$

we consider that $lim_{x\to\infty}\phi_F(x) = \infty$. Similarly, $lim_{x\to-\infty}\phi_F(x) = -\infty$. Hence, $\phi_F(x)$ is not bounded.

iii

When $J(t) = (\beta - \alpha)^{-1} I_{(\alpha,\beta)}(t)$,

$$\phi_F(x) = -\frac{1}{\beta - \alpha} \int_{F^{-1}(\alpha)}^{F^{-1}(\beta)} |\delta_x(y) - F(y)| dy \leq \frac{F^{-1}(\beta) - F^{-1}(\alpha)}{\beta - \alpha}.$$

Problem 3

Suppose that $\alpha = \beta(P_1)$. Then the test$T_0 \equiv \alpha$ is also a UMP test by definition. By the uniqueness of the UMP test, we must have $f_1(x) = cf_0(x)$ a.e. ν, which implies $c = 1$. Therefore, $f_1(x) = f_0(x)$ a.e ν, i.e.,$P_0 = P_1$.

Problem 4

i

Let $X_{(1)}$ be the smallest order statistic. Since

$$\frac{f_{\theta_1}(X)}{f_{\theta_0}(X)} = \begin{cases} e^{n(\theta_1 - \theta_0)}, X_{(1)} > \theta_1 \\ 0, \theta_0 < X_{(1)} \leq \theta_1 \end{cases}$$

the UMP test is either

$$T_1 = \begin{cases} 1, X_{(1)} > \theta_1 \\ \gamma, \theta_0 < X_{(1)} \leq \theta_1 \end{cases}$$

or

$$T_2 = \begin{cases} \gamma, X_{(1)} > \theta_1 \\ 0, \theta_0 < X_{(1)} \leq \theta_1 \end{cases}$$

When $\theta = \theta_0, P(X_{(1)} > \theta_1) = e^{n(\theta_1 - \theta_0)}$. If $e^{n(\theta_1 - \theta_0)} \leq \alpha$, then T_1 is the UMP test since under$\theta = \theta_0$,

$$\begin{aligned} E(T_1) &= P(X_{(1)} > \theta_1) + \gamma P(\theta_0 < X_{(1)} \leq \theta_1) \\ &= e^{n(\theta_1 - \theta_0)} + \gamma(1 - e^{n(\theta_1 - \theta_0)}) = \alpha \end{aligned}$$

with $\gamma = \dfrac{\alpha - e^{n(\theta_1 - \theta_0)}}{1 - e^{n(\theta_1 - \theta_0)}}$. If$e^{n(\theta_1 - \theta_0)} > \alpha$, then T_2 is the UMP test since under null,

$$E(T_2) = \gamma P(X_{(1)} > \theta_1) = \gamma e^{n(\theta_1 - \theta_0)}$$

with $\gamma = \dfrac{\alpha}{e^{n(\theta_1 - \theta_0)}}$.

ii

Suppose $\theta_1 > \theta_0$. Then

$$\frac{f_{\theta_1}(X)}{f_{\theta_0}(X)} = \begin{cases} \dfrac{\theta_1^n}{\theta_0^1}, X_{(1)} > \theta_1 \\ 0, \theta_0 < X_{(1)} \leq \theta_1 \end{cases}$$

The UMP test is either

$$T_1 = \begin{cases} 1, X_{(1)} > \theta_1 \\ \gamma, \theta_0 < X_{(1)} \leq \theta_1 \end{cases}$$

or

$$T_2 = \begin{cases} \gamma, X_{(1)} > \theta_1 \\ 0, \theta_0 < X_{(1)} \leq \theta_1 \end{cases}$$

When $\theta = \theta_0, P(X_{(1)} > \theta_1) = \frac{\theta_0^n}{\theta_1^n}$. If $\frac{\theta_0^n}{\theta_1^n} \leq \alpha$, then T_1 is UMP test since under null,

$$E(T_1) = \frac{\theta_0^n}{\theta_1^n} + \gamma(1 - \frac{\theta_0^n}{\theta_1^n}) = \alpha$$

with $\gamma = \frac{\alpha - \frac{\theta_0^n}{\theta_1^n}}{1 - \frac{\theta_0^n}{\theta_1^n}}$. If $\frac{\theta_0^n}{\theta_1^n} > \alpha$, then T_2 is the UMP test since under null

$$E(T_2) = \gamma\frac{\theta_0^n}{\theta_1^n} = \alpha$$

Suppose now that $\theta_1 < \theta_0$. Then

$$\frac{f_{\theta_1}(X)}{f_{\theta_0}(X)} = \begin{cases} \frac{\theta_1^n}{\theta_0^1}, X_{(1)} > \theta_1 \\ \infty, \theta_0 < X_{(1)} \leq \theta_1 \end{cases}$$

The UMP test is either

$$T_1 = \begin{cases} 0, X_{(1)} > \theta_1 \\ \gamma, \theta_0 < X_{(1)} \leq \theta_1 \end{cases}$$

or

$$T_2 = \begin{cases} \gamma, X_{(1)} > \theta_1 \\ 1, \theta_0 < X_{(1)} \leq \theta_1 \end{cases}$$

When $\theta = \theta_0, E(T_1) = 0$ and $E(T_2) = \gamma$. Hence the UMP test is T_2.