Homework 11

Problem 1(#6.17)

i

Let $f(x)$ and $g(x)$ be the Randon-Nikodym derivatives of $F(x)$ and $G(x)$ with respect to the measure ν induced by $F(x) + G(x)$, respectively. The probability density of X is $\theta f(x) + (1-\theta)g(x)$. For $0 \leq \theta_1 \leq \theta_2 \leq 1$,

$$\frac{\theta_2 f(x) + (1-\theta_2)g(x)}{\theta_1 f(x) + (1-\theta_1)g(x)} = \frac{\theta_2 \frac{f(x)}{g(x)} + (1-\theta_2)}{\theta_1 \frac{f(x)}{g(x)} + (1-\theta_1)}$$

is nondecreasing in $Y(x) = f(x)/g(x)$. Hence, the family of densities of X has monotone likelihood ratio in $Y(X) = f(X)/g(X)$ and a UMP test is given as

$$T = \begin{cases} 1, Y(X) > c \\ \gamma, Y(X) = c \\ 0, Y(X) < c \end{cases}$$

where c and γ are uniquely determined by $E(T(X)) = \alpha$ when $\theta = \theta_0$.

ii

For any test T, its power is

$$\beta_T(\theta) = \int T(x)[\theta f(x) + (1-\theta)g(x)]d\nu = \theta \int T(x)[f(x) - g(x)]d\nu + \int T(x)g(x)du$$

which is a linear function of θ on [0,1]. If T has level α, then $\beta_T(\theta) \leq \alpha$ for any $\theta \in [0,1]$. Since the power T_* is equal to the constant α, we conclude that T_* is a UMP test of size α.

(#6.29) i

Let $\beta_T(\theta)$ be the power function of a test T. For any test T of level α such that $\beta_T(\theta)$ is not constant, either $\beta_T(0)$ or $\beta_T(1)$ is strictly less than α. Without loss of generality, assume that $\beta_T(0) < \alpha$. This means that at $\theta = 0$, which is one of parameter values under H_1, the power of T is smaller than $T_* \equiv \alpha$. Hence, any T with nonconstant power function can not be UMP. From Exercise 12, the UMP test of size α for testing $H_0 : \theta \leq \theta_1$ versus $H_1 : \theta > \theta_1$ clearly has power larger than α at $\theta = 1$. Hence, $T_* \equiv \alpha$ is not UMP. Therefore,a UMP test does not exist.

ii

If a test T of level α has a nonconstant power function, then either $\beta_T(0)$ or $\beta_T(1)$ is strictly less than α and, hence, T is not unbiased. Therefore, only tests with constant power functions may be unbiased. This implies that $T_* \equiv \alpha$ is a UMPU test of size α.

Problem 2(# 6.39)

i

When $\mu = 0, 1, ..., n_1 + n_2$ amd $y \in A$,

$$P(Y = y, U = u) = \binom{n_1}{\mu - y}\binom{n_2}{y} p_1^{\mu-y}(1-p_1)^{n_1-\mu+y} p_2^y (1-p_2)^{n_2-y}$$

and

$$P(U = \mu) = \sum_{y \in A} \binom{n_1}{\mu - y}\binom{n_2}{y} p_1^{\mu-y}(1-p_1)^{n_1-\mu+y} p_2^y (1-p_2)^{n_2-y}.$$

Then, when $y \in A$,

$$P(Y = y|U = \mu) = \frac{P(Y = y, U = u)}{P(U = u)} = \binom{n_1}{\mu - y}\binom{n_2}{y} e^{\theta y} K_\mu(\theta).$$

ii

Since $\theta = log(\frac{p_2(1-p_1)}{p_1(1-p_2)})$, the testing problem is equivalent to testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$. By theorem 6.4 in Shao (2003), the UMPU test is

$$T_*(Y, U) = \begin{cases} 1, Y > C(U) \\ \gamma(U), Y = c(U) \\ 0, Y < C(U) \end{cases}$$

where C and γ are functions of U such that $E(T_*|U) = \alpha$ when $\theta = 0$, which can be determined using the conditional distribution of Y given U. when $\theta = 0$, this conditional distribution is, by the result in (1),

$$P(Y = y|U = u) = \binom{n_1 + n_2}{\mu}^{-1}\binom{n_1}{\mu - y}\binom{n_2}{y} I_A(y), \mu = 0, 1, ..., n_1 + n_2.$$

iii

The testing problem is equivalent to testing $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$. Thus, the UMPU test is

$$T_* = \begin{cases} 1, Y > C_1(U) or Y < C_2(U) \\ \gamma_i(U), Y = c_i(U), i = 1, 2 \\ 0, C_1(U) < Y < C_2(U) \end{cases}$$

where C_i's and γ_i's are functions such that $E(T_*|U) = \alpha$ and $E(T_*Y|U) = \alpha E(Y|U)$ when $\theta = 0$, which can be determined using the conditional distribution of Y given U in part (2) of the solution.

Problem 3(# 6.49)

Let $Y = \bar{X}_2 - \bar{X}_1$, $U_1 = n_1\bar{X}_1 + n_2\bar{X}_2$, $U_2 = \sum_{i=1}^{2}\sum_{j=1}^{n} X_{ij}^2$, $\theta = (\mu_1 - \mu_2)/[(n_1^{-1} + n_2^{-1})\sigma^2]$, $\varphi_1 = (n_1\mu_1 + n_2\mu_2)/[(n_1 + n_2)\sigma^2]$, and $\varphi_2 = -(2\sigma^2)^-1$. Then, the joint density of $X_{i1}, ..., X_{in_i}$, $i = 1, 2$, can be written as

$$(\sqrt{2\pi}\sigma)^{n_1+n_2}e^{\theta Y+\varphi_1 U_1+\varphi_2 U_2}.$$

The statistic $V = Y/\sqrt{U_2 - U_1^2/(n_1 + n_2)}$ satisfies the conditions in Lemma 6.7(ii) in Shao. Hence, the UMPU test has the rejection region $V < c_1$ or $V > c_2$. Under H_0, V is symmetrically distributed around 0, i.e., V and -V have the same distribution. Thus, a UMPU test rejects H_0 when $-V < c_1$ or $-V > c_2$, which is the same as rejecting H_0 when $V < -c_2$ or $V > -c_1$. By the uniqueness of the UMPU test, we conclude that $c_1 = -c_2$, i.e., the UMPU test rejects when $|V| > c$. Since

$$U_2 - \frac{U_1^2}{n_1 + n_2} = (n_1 - 1)S_1^2 + (n_2 - 1)S_2^2 + \frac{n_1 n_2 Y^2}{n_1 + n_2}$$

we obtain that

$$\frac{1}{V^2} = \frac{n_1 n_2}{(n_1 + n_2)(n_1 + n_2 - 2)}\frac{1}{[t(X)]^2} + \frac{n_1 n_2}{n_1 + n_2}$$

Hence, $|V|$ is an increasing function of $|t(X)|$. Also, $t(X)$ has the t-distribution with $t_{n_1+n_2-2}$ under H_0. Thus, the UMPU test rejects H_0 when $|t(X)| > t_{n_1+n_2-2,\alpha/2}$. Under H_1, $t(X)$ is distributed as the noncentral t-distribution $t_{n_1+n_2-2}(\delta)$ with noncentrality parameter

$$\delta = \frac{\mu_2 - \mu_1}{\sigma\sqrt{n_1^{-1} + n_2^{-1}}}$$

Thus the power function of the UMPU test is

$$1 - G_\delta(t_{n_1+n_2-2,\alpha/2}) + G_\delta(-t_{n_1+n_2-2,\alpha/2}),$$

Where G_δ denotes the cumulative distribution function of the noncentral t-distribution $t_{n_1+n_2-2}(\delta)$

Problem 4(# 6.14(a))

The family of densities has monotone likelihood ratio in $T(X) = \sum_{i=1}^{n} X_i$, which has the gamma distribution with shape parameter n and scale parameter θ. Under H_0, $2T/\theta_0$ has the chi-square distribution χ_{2n}^2. Hence, the UMP test is

$$T_*(X) = \begin{cases} 1, T(X) > \theta_0\chi_{2n,\alpha}^2/2 \\ 0, T(X) \le \theta_0\chi_{2n,\alpha}^2/2 \end{cases}$$

where $\chi_{r,\alpha}^2$ is the $(1 - \alpha)$th quantile of the chi-square distribution χ_r^2.