Homework 2

1 Prob 1

(b) Define $\hat{\mu}_1 = \bar{X}$ and $\hat{\mu}_2 = n^{-1}\sum_{i=1}^n X_i^2$.

$$EX_1 = E[E(X_1|p)] = E(kp) = \frac{k\alpha}{\alpha+\beta}$$

and

$$EX_1^2 = E[E(X_1^2|p)] = E[kp(1-p)+k^2p^2] = \frac{k\alpha}{\alpha+\beta} + \frac{(k^2-k)\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

, setting $\hat{\mu_1} = EX_1$ and $\hat{\mu_2} = EX_1^2$,we obtain that

$$\hat{\alpha} = \frac{\hat{\mu_2} - \hat{\mu_1} - \hat{\mu_1}(k-1)}{\hat{\mu_1}(k-1) + k(1-\hat{\mu_2}/\hat{\mu_1})}$$

and $\hat{\beta} = \frac{k\hat{\alpha}}{\hat{\mu_1}} - \hat{\alpha}$, then the empirical Bayes action is $(n\bar{X}+\hat{\alpha})/(kn+\hat{\alpha}+\hat{\beta})$
(c) $EX_1 = E[E(X_1|\theta)] = E(\theta/2) = \frac{ab}{2(b-1)}$ and $EX_1^2 = E[E(X_1^2|\theta)] = E[(\theta/2)^2 + \theta^2/12] = E[\theta^2/3] = \frac{a^2b}{3(b-2)}$, setting $\hat{\mu_1} = E(X_1)$ and $\hat{\mu_2} = E(X_1^2)$, we obtain that

$$\hat{b} = 1 + \sqrt{3\hat{\mu_2}/(3\hat{\mu_2} - 4\hat{\mu_1}^2)}$$

and

$$\hat{a} = 2\hat{\mu_1}(\hat{b}-1)/\hat{b}.$$

Therefore, the empirical Bayes action is $(n+b)max(X_{(n)},\hat{a})/(n+\hat{b}-1)$ where $X_{(n)}$ is the largest order statistic.
(d) $EX_1 = E[E(X_1|\theta)] = E(\theta) = \frac{1}{\gamma(\alpha-1)}$, and $EX_1^2 = E[E(X_1^2|\theta)] = E(2\theta^2) = \frac{2}{\gamma^2(\alpha-1)(\alpha-2)}$. Setting $\hat{\mu_1} = E(X_1)$ and $\hat{\mu_2} = E(X_1^2)$, we obtain that

$$\hat{\alpha} = \frac{2\hat{\mu_2} - 2\hat{\mu_1^2}}{\hat{\mu_2} - 2\hat{\mu_1^2}}$$

and $\hat{\gamma} = \frac{1}{(\hat{\alpha}-1)\hat{\mu_1}}$, so the empirical Bayes action is $(\hat{\gamma}n\bar{X}+1)/[\hat{\gamma}(n+\hat{\alpha}-1)]$.

2 Prob 2

2.1

Note that $EX_1 = E[E(X_1|\theta)] = E(\theta/2) = ab/[2(b-1)]$. Then $\hat{\alpha} = \frac{2(b-1)}{bn}\sum_{i=1}^n X_i$ is the moment estimator of a. From question 1 in HW1, the empirical Bayes action is $\delta(\hat{a})$.

2.2

The joint density for (X, θ, a) is

$$ba^{b-1}\theta^{-(n+b+1)}I_{(X_{(n)},\infty)}(\theta)I_{(0,\theta)}(a)$$

Hence, the joint density for (X, θ) is

$$\int_0^\theta ba^{b-1}\theta^{-n-b-1}I_{(X_{(n)},\infty)}(\theta)da = \theta^{-n-1}I_{(X_{(n)},\infty)}(\theta)$$

and the generalized Bayes action is

$$\frac{\int_{X_{(n)}}^\infty \theta^{-n}d\theta}{\int_{X_{(n)}}^\infty \theta^{-n-1}d\theta} = \frac{nX_{(n)}}{n-1}$$

3 Prob 3

3.1

For a given θ, T has the gamma distribution with shape parameter θ and scale parameter 1. Hence, the joint probability density of (T, θ) is

$$f(t,\theta) = \frac{1}{(\theta-1)!}t^{\theta-1}e^{-t}p(1-p)^{\theta-1}$$

and

$$f(t) = \sum_{\theta=1}^\infty f(t,\theta) = pe^{-t}\sum_{\theta=1}^\infty \frac{[(1-p)t]^{\theta-1}}{(\theta-1)!} = pe^{-pt}.$$

Then,

$$E[L(\theta,a)|T=t] = p^{-1}e^{pt}\sum_{\theta=1}^\infty \frac{(\theta-a)^2}{\theta}f(t,\theta) = e^{-\xi}\sum_{\theta=1}^\infty \frac{\xi^{\theta-1}}{\theta!}\theta^2 - 2ae^{-\xi}\sum_{\theta=1}^\infty \frac{\xi^{\theta-1}}{\theta!}\theta + a^2e^{-\xi}\sum_{\theta=1}^\infty \frac{\xi^{\theta-1}}{\theta!}$$

$$= 1 + \xi - 2a + (1 - e^{-\xi}a^2)/\xi$$

3.2

Since $E[L(\theta, a)|T = t]$ is a quadratic function of a, the Bayes estimator is $\delta(T) = (1-p)T/(1-e^{-(1-p)T})$. The posterior expected loss when T=t is

$$E[L(\theta,\xi(t))|T=t] = 1 - \frac{\xi e^{-\xi}}{1-e^{-\xi}} = 1 - \xi\sum_{m=1}^\infty e^{-m\xi}$$

3.3

As shown in part (i) of the solution, the marginal density of T is $\sum_{\theta=1}^\infty f(t,\theta) = pe^{-pt}$, which is the density of the exponential distribution with scale parameter p^{-1}.

3.4

The Bayes risk of $\delta(T)$ is

$$E[E[L(\theta,\delta(T))|T]] = 1_E[(1-p)T\sum_{m=1}^{\infty} e^{-m(1-p)T}] = 1-(1-p)p\sum_{m=1}^{\infty}\int_{m=1}^{\infty} te^{-m(1-p)t}e^{-pt}dt$$

$$= 1-(1-p)p\sum_{m=1}^{\infty}\frac{1}{[m(1-p)+p]^2}$$

, where the first equality follows from the result in (ii) and the second equality follows from the result in (iii)

4 Prob 4

4.1

assume $\delta(x)$ is a location invariant estimator. Denote the location parameter to be θ, the variance is

$$Var_\theta(\delta(X)) = E_\theta(\delta(X)^2) - (E_\theta(\delta(X)))^2 = E_0(\delta(X=\theta)^2) - (E_0(\delta(X+\theta)))^2 = Var_0(\delta(X))$$

where the second equality follows by the invariance of the family and the third follows by the equivariance of the estimator. Thus the variance is independent of θ.

4.2

The risk is

$$E_\theta(L(\theta,\delta(X))) = E_0(L(\theta,\delta(X+\theta))) = E_0(L(\theta,\delta(X)+\theta)) = E_0(L(0,\delta(X)))$$

where the first equality follows by the invariance of thr family, the second by the equivariance of the estimator, and the third by the invariance of the loss function. Thus the risk is independent of θ

5 Prob 5

5.1

By problem 4 in HW2, the risk $R(\delta,\theta) = E_0(L(\delta))$, let $\delta = \delta_0 - u(D)$ it becomes

$$E_0(L(\delta_0(x)-u(D))) = E(E_0(L(\delta_0(x)-u(D))|D=d)) = E_0(L(\delta_0(x))-u(d))$$

, for any d. Since $R(\delta,\theta)$ is constant, we have thayt $u(d)$ is constant. So $R(\delta,\theta) = E_0(L(\delta_0(x))-u)$ if $u^* = argmin_u E_0(L(\delta_0(x))-u)$, then $\delta^* = \delta_0 - u^*$ minimizes the risk $R(\delta,\theta)$, By thm 4.3, δ^* is location invariant. So δ^* is MRIE. u^* is constant and independent of the value of D.

5.2

5.3

Let $D = (X_1 - X_n, ..., X_{n-1} - X_n)$. Then the distribution of D does not depend on θ. Since $X_{(1)}$ and D are independent and $X_{(1)}$ is location invariant,$X_{(1)} - u^*$ is an MRIE of θ, where u^* minimizes $E_0[L(X_{(1)} - \mu)]$ over μ and E_0 is the expectation taken under $\theta = 0$. since $E_0[L(X_{(1)} - \mu)] = E_0|X_{(1)} - \mu|$, u^* is the median of the distribution of $X_{(1)}$ when $\theta = 0$. Since $X_{(1)}$ has Lebesgue density $ne^{-nx}I_{(0,\infty)}$ when $\theta = 0$,

$$\frac{1}{2} = n\int_0^{u^*} e^{-nx}dx = 1 - e^{-nu^*}$$

and hence $u^* = log2/n$.