5061 Homework 3

1 Problem 1

1.1 prop 4.5

Since T_0 is a scale invariant estimator of σ^h, for any $r > 0$, $T_0(rx) = r^h T_0(x)$. Thus, if exists μ on R^n s.t. $T(x) = T_0(x)/\mu(z)$ for all $x \in R^n$ where $z = (z_1, ..., z_n)$, $z_i = x_i/x_n$, $i = 1, ..., n-1$, $z_n = x_n/|x_n|$, then we have for any $r > 0$,

$$u(rx) = u(\frac{rx_1}{rx_n}, \frac{rx_2}{rx_n}, ..., \frac{rx_n}{r|x_n|}) = u(\frac{x_1}{x_n}, ..., \frac{x_n}{|x_n|}) = u(x)$$

$$T(rx) = \frac{T_0(rx)}{u(rx)} = \frac{r_h T_0(x)}{u(x)} = r^h T(x)$$

Thus, T is scale invariant. If T is scale invariant, let $u(x) = \frac{T_0(x)}{T(x)}$, then $u = \frac{|x_n|^h T_0(\frac{x_1}{|x_n|}, ..., \frac{x_n}{|x_n|})}{|x_n|^h T(\frac{x_1}{|x_n|}, ..., \frac{x_n}{|x_n|})}$

$$= \frac{T_0(z_1 sign(x_n), ..., z_{n-1} sign(x_n), z_n)}{T_1(z_1 sign(x_n), ..., z_{n-1} sign(x_n), z_n)}$$

$$= \frac{T_0(z_1 sign(z_n), ..., z_{n-1} sign(z_n), z_n)}{T_1(z_1 sign(z_n), ..., z_{n-1} sign(z_n), z_n)}$$

we find a μ that depends on X only through z.

1.2 Theorem 4.7

$R(\theta, \delta) = E_\theta[L(\theta, \delta(x))] = E_\theta[L(\bar{g}\theta, \tilde{g}\delta(x))]$ since loss function is invariant.

$$= E_\theta[L(\bar{g}\theta, \delta(g(x)))]$$

$$= E_{\theta^*}[L(\theta^*, \delta(x^*))], gx = x^* \in P$$

$$= R(\theta^*, \delta)$$

$$= R(\bar{g}\theta, \delta)$$

thus the risk function of δ is a constant.

1.3 Theorem 4.8

1.By prop 4.5, Let $T^*(x) = T_0(x)/u(z)$, $T^*(x)$ is scale invariant. By theorem 4.7,

$$R(T^*, \sigma) = R(T^*, 1) = E_1(L(T^*(x)))$$

$$= E_1[L(T_0(x)/\mu(z))]$$

$$= E[E_1[L(T_0(x)/\mu(z))|Z = z]]$$

$R(T^*, \sigma)$ can be minimized if $E_1[L(T_0(x)/\mu(z))|Z = z]$ is minimized. So if exists a $\mu^*(z)$ s.t. $\mu^*(z) = argmin_\mu E_1[L(T_0(x)/\mu(z))|Z = z]$ for each z, then $T^*(x) = T_0(X)/u^*(Z)$ is MRIE.
2.$r(t) = L(e^t)$ by replacing $L(w)$ by $r(log(w))$ with $r(-\infty) = L(0)$.

$$r(log(w)) = L(e^{log(w)}) = L(w)$$

$$u^* = argmin_\mu E_1[L(T_0(x)/u(z))|Z = z]$$

$$= argmin_\mu E[r(log\frac{T_0(x)}{\mu(x)})|Z = z]$$

$$= argmin_\mu E[r(log(T_0(x)) - log(\mu(z)))|Z = z]$$

This results reduce to theorem 4.5(ii), thus u^* exists if $r(t) = L(e^t)$ is convex and not monotone. It is unique if $r(t)$ is strictly convex.

1.4 Cor 4.1

$R(\sigma^2, \delta) = R(1, \delta)$ since risk function is constant.

$$= E_1[L(1, \delta)] = E_1[(\delta - 1)^2] = E_1[(T_0(x)/u(z) - 1)^2]$$

$$= E[E_1[(T_0(x)/u(z) - 1)^2]|Z = z]$$

By taking derivative of $R(\sigma^2, \delta)$ we get $\delta(x) = T^*(x) = T_0(x)/u^*(z) = \dfrac{T_0(x)E_1(T_0(x)|z)}{E_1(T_0^2(x)|z)}$ Consider $T_0(x) = |x_n|^h$ and denote $z = (z_0, s)$ where $z_0 = (\frac{x_1}{x_n}, ..., \frac{x_{n-1}}{x_n}) = (z_1^0, \ldots, z_{n-1}^0)$, $s = \frac{x_n}{|x_n|}$

$$P(|x_n| \leqslant t|z_0, s) = \int_{-t}^{t} \frac{f(x_n, z_0, s)}{\int_{-\infty}^{\infty} f(x_n, z_0, s)dx_n} dx_n$$

$$= \frac{\int_{-t}^{t} f(x_n, z_0)f(s|x_n)dx_n}{\int_{-\infty}^{\infty} f(x_n, z_0)f(s|x_n)dx_n}$$

where $f(s|x_n) = 1(x_n > 0)$ if s=1 and $f(s|x_n) = 1(x_n < 0)$ if s=-1. Suppose $s = 1$ then $P(|x_n| \leqslant t|z_0, s) = \frac{\int_0^t f(x_n,z_0)dx_n}{\int_0^\infty f(x_n,z_0)dx_n}$ Thus$f(|x_n||z_0, s = 1) = \frac{f(x_n,z_0)}{\int_0^\infty f(x_n,z_0)dx_n}$ under transformation $z = (z_0, s)$

$$f(z_0, x_n) = f(z_1^0, \ldots, z_{n-1}^0, x_n) = f_X(z_1^0 x_n, \ldots, z_{n-1}^0 x_n, x_n)|x_n|^{n-1}$$

$$f(x_n|z_0, s = 1) = \frac{f_X(z_1 t, \ldots, z_{n-1} t, t)t^{n-1}}{\int_0^\infty f_X(z_1 t, \ldots, z_{n-1} t, t)t^{n-1} dt}$$

let $t = x_n v, z_1 = \frac{x_r}{x_n}$

$$E_1[|x_n^h|z_0, s=1] = \frac{\int_0^\infty t^h f(z_1 t, ..., z_{n-1}t, t)t^{n-1}dt}{\int_0^\infty f(z_1 t, ..., z_{n-1}t, t)t^{n-1}dt} = \frac{\int_0^\infty x_n^h v^h f(z_1 x_n v, ..., z_{n-1}x_n v, x_n v)x_n^{n-1}v^{n-1}|x_n|dv}{\int_0^\infty f(z_1 x_n v, ..., z_{n-1}x_n v, x_n v)x_n^{n-1}v^{n-1}|x_n|dv}$$

$$= \frac{\int_0^\infty |x_n|^h v^{h+n-1} f(x_1 v, ..., x_{n-1}v, x_n v)dv}{\int_0^\infty f(x_1 v, ..., x_{n-1}v, x_n v)dv}$$

we have the same result for s=-1, also let $v = t$, $T^*(x) = \frac{|x_n|^h \int_0^\infty |x_n|^h v^{h+n-1} f(x_1 v, ..., x_n v)dv}{\int_0^\infty |x_n|^{2h} v^{2h+n-1} f(x_1 v, ..., x_n v)dv} = \frac{\int_0^\infty t^{h+n-1} f(x_1 v, ..., x_n v)dt}{\int_0^\infty t^{2h+n-1} f(x_1 v, ..., x_n v)dt}$

2 Problem 2

That any such loss function is invariant is obvious. Conversely, suppose that L is invariant and that $(\mu, \sigma; a)$ and $(\mu', \sigma'; a')$ are two points with $(a' - \mu')/\sigma' = (a - \mu)/\sigma$. Putting $r = \sigma'/\sigma$ and $\mu' - c = r\mu$ one has $a' = c + ra$, $\mu' = c + r\mu$ and $\sigma' = r\sigma$, hence $L(\mu', \sigma'; a') = L(\mu, \sigma; a)$, as was to be proved.

3 Problem 3

Transformation:$g(x, y) = (rx, r'y) = (u, v)$ where $r > 0, r' > 0$ $f_{u,v}(u, v) = f_{x,y}(\frac{u}{r}, \frac{v}{r'})\frac{1}{r^m}\frac{1}{r'^n}$, let $\sigma_x^* = r\sigma_x, \sigma_y^* = r'\sigma_y$, then $f_{u,v}(u, v) = \frac{1}{\sigma_x^{*m}\sigma_y^{*n}} f(u_1/\sigma_x^*, ..., u_m/\sigma_x^*, v_1/\sigma_y^*, ..., v_n/\sigma_y^*)$, $\theta = (\sigma_x, \sigma_y)$,$\theta^* = (\sigma_x^*, \sigma_y^*) = \bar{g}\theta$. $(x, y) \sim P_\theta, g(x, y) \sim p_{\theta^*}, P_{\theta^*} \in P$, for any $\theta^* \in \Theta$ exists θ such that $\theta^* = c\theta$, Hence P is invariant to g.

$$L(\bar{g}\theta, \theta^*) = L(\frac{a^*}{\eta^*}) = L(\frac{a^*}{(r'/r)^h \eta}) = L(a/\eta) = L(\theta, a)$$

where $a^* = (r'/r)^h a = \tilde{g}a$.

4 problem 4

(a) $P(X \leqslant x) = P(\epsilon \leqslant x - z\beta)$, $f_x(x) = \sigma^{-n}\prod_{i=1}^n f(\frac{x_i - z_i'\beta}{\sigma})$, then $P(g(x) \leqslant x) = P(\epsilon \leqslant (x - zc)/r - z\beta)$, $f_g(x) = (r\sigma)^{-n}\prod_{i=1}^n f(\frac{x_i - z_i'c - z_i'\beta r}{r\sigma})$, where $r > 0, c \in R^p$, thus $\bar{g}(\beta, \sigma) = (c + r\beta, r\sigma)$, P is invariant under g.
(b) Only nee dto show the loss function is invariant. $L(\frac{a - l'\beta}{\sigma}) = L(\theta, a)$

$$L(\bar{g}\theta, a^*) = L((c + r\beta, r\sigma), a^*) = L(\frac{a^* - l'(c + r\beta)}{r\sigma}) = L(\frac{a - l\beta}{\sigma}) = L(\theta, a)$$

, where $a^* = ra + l'c = \tilde{g}(a)$
(c)$\hat{\beta} = (Z'Z)^{-1}Z'X, T(X) = l'\hat{\beta} = l'(Z'Z)^{-1}Z'X$

$$T(g(X)) = T(rX + Zc) = l'(Z'Z)^{-1}Z'(rX + Zc) = l'(Z'Z)^{-1}Z'Xr + l'c$$

$$\tilde{g}T(X) = rT(X) + l'c = T(g(x))$$

(d)Theorem 4.10

(i) By theorem 4.9, an MRIE of $l'\beta$ is $T^*(X) = T_0(X) - u^*(W)T_1(X)$,where W is defined in (4.28),$T_0(X) = l'\hat{\beta}.T_1(X)$ is an invariant estimator for σ. And $u^*(w) = \arg\min E_{01}[L(T_0(x) - u(w)T_1(x))|W = w] = \arg\min E_{01}[L(T_0(x) - u(w)T_1(x))]$ because w is an ancillary statistic. And since L is convex and even, we have that $u^*(w) = 0$.

(ii) $L(a,\sigma) = \dfrac{(a^2-\sigma^2)^2}{\sigma^4}$, $T(X) = \frac{SSR}{n-r+2} = var(l'\hat{\beta}) = l'(z'z)^{-1}z'var(\varepsilon)z(z'z)^{-1}l$, $T(g(X)) = var(l'(c + r\hat{\beta})) = r^2 var(l'\hat{\beta})$ when $\tilde{g}a = a^* = r^2 a$. We have $L(a^*,\sigma^*) = L(a^*, r\sigma) = \dfrac{(a^* - r^2\sigma^2)^2}{r^4\sigma^4} = \dfrac{(a-\sigma^2)^2}{\sigma^4} = L(a,\sigma)$. $\tilde{g}(T(X)) = r^2 Var(l'\hat{\beta}) = T(g(X))$. So $T(X)$ is invariant. So $T(X)$ is MRIE for σ^2.