Homework 4

Problem 1

(a)

$\mathcal{X} = \{0, 1, 2, \ldots, n\}$.

(b)

$\Theta = (0, 1)$.

(c)

$G = \{g : g(x) = n - x, x \in \mathcal{X}\}$.

(d)

$\bar{G} = \{\bar{g}(p) = 1 - p, p \in (0, 1)\}$.

(e)

The orbits are the pairs $\{p, 1-p\}, p \in (0, 1)$. The group is not transitive.

Problem 2

(a)

Let $A = \begin{pmatrix} c_1 & c_2 \\ 0 & b \end{pmatrix}$, $\Sigma^* = A\Sigma A'$ and $\theta^* = d^2\theta$, then $g_1(X_1, X_2, Y_1, Y_2) \sim N_4\left(0, \begin{pmatrix} \Sigma^* & 0 \\ 0 & \theta^*\Sigma^* \end{pmatrix}\right)$ and $\bar{g_1}(\theta, \Sigma) = (d^2\theta, A\Sigma A')$.

(b)

Let $a^* = d^2 a$.
$(\Rightarrow)$ $\bar{g_1}(\theta) = d^2\theta$ is from a scale family. A loss function is scale invariant only when $L(\theta, a) = L(\frac{a}{\theta})$;
$(\Leftarrow)$ $L(\theta, a) = L(\frac{a}{\theta}) = L(\frac{d^2 a}{d^2\theta}) = L(d^2\theta, d^2 a) = L(\bar{g_1}(\theta), a^*)$.

(c)

($\Leftarrow$) $\delta(g_1(x_1,x_2,y_1,y_2)) = k\dfrac{(dby_2)^2}{(bx_2)^2} = d^2\delta(x_1,x_2,y_1,y_2) = \tilde{g}_1(\delta(x_1,x_2,y_1,y_2))$
($\Rightarrow$) Let $G_1' = \{g_1 : c_1, c_2 > 0, b = d = 1\}$.
$g_1(x_1,x_2,y_1,y_2) = (c_1x_1 + c_2x_2, x_2, c_1y_1 + c_2y_2, y_2)$ for $g_1 \in G$.
δ is invariant $\Rightarrow$ $\delta(x_1^*, x_2^*, y_1^*, y_2^*) = d^2\delta(x_1,x_2,y_1,y_2) = \delta(x_1,x_2,y_1,y_2) \Rightarrow \delta$ doesn't depend on x_1, y_1.
$\delta(bx_2, dby_2) = d^2\delta(x_2,y_2) \Rightarrow \delta(x_2,y_2) = k\dfrac{y_2^2}{x_2^2}$ for some value of K (a.e.)

(d)

$R(\theta,\delta) = E[L(\theta,\delta)] = E\left[L\left(\dfrac{\delta}{\theta}\right)\right] = E\left[L\left(\dfrac{ky_2^2}{\theta x_2^2}\right)\right]$
$R(1,\delta) = E_1[L(1,\delta)] = E_1\left[L\left(\dfrac{\delta}{1}\right)\right] = E_1\left[L\left(\dfrac{ky_2^2}{x_2^2}\right)\right]$
$R(\theta,\delta) = R(1,\delta)$
Thus $k^* = \arg\min E_\theta\left[L\left(\dfrac{ky_2^2}{\theta x_2^2}\right)\right] = \arg\min E_1\left[L\left(\dfrac{ky_2^2}{x_2^2}\right)\right]$.

(e)

$x_1^* = bx_1, y_1^* = dby_1, x_2^* = c_1x_1 + c_2x_2, y_1^* = dby_1, y_2^* = d(c_1y_1 + c_2y_2)$.
MRIE: $\delta^* = k^*\dfrac{y_1^2}{x_1^2}$ where $k^* = \arg\min E_\theta\left[L\left(\dfrac{ky_1^2}{\theta x_1^2}\right)\right] = \arg\min E_1\left[L\left(\dfrac{ky_1^2}{x_1^2}\right)\right]$

Problem 3

Let δ be MRIE. Since $\bar{G}$ is transitive. $\exists \bar{g} \in \bar{G}$ s.t. $\bar{g}(\theta^*) = \theta$. Since $\tilde{G}$ is commutative for $h \in G$. $\tilde{h}\tilde{g}\delta = \tilde{g}\delta h \Rightarrow \tilde{g}\delta$ is invariant. $E_\theta[L(\theta^*, \delta(X))] = E_\theta[L(\bar{g}(\theta^*)), \tilde{g}(\delta(X))] = E_\theta[L(\theta, \tilde{g}(\delta(X)))] \geq E_\theta[L(\theta, \delta(X))]$.

Problem 4

(a)

$\bar{g}_0(\mu,\sigma) = (c_0\mu + b_0, c_0\sigma), \bar{g} \circ \bar{g}_0 = (c(c_0\mu + b_0) + b, cc_0\sigma)$.
Let $c' = cc_0, b' = cb_0 + b$. $\mu_r(A \times B) = \int_A \int_B \dfrac{1}{c} \mathrm{d}b\mathrm{d}c = \int_A' \int_B' \dfrac{1}{c'} \mathrm{d}b'\mathrm{d}c' = \mu_r(A' \times B')$.

(b)

$\mu | \bar{X}, \sigma^2 \sim N(\bar{X}, \dfrac{\sigma^2}{n})$
$\sigma^{-2} | \bar{X} \sim \text{Gamma}(\dfrac{n}{2} + 1, \dfrac{1}{2}\sum_{i=1}^n (X_i - \bar{X})^2)$

(c)

Let $T_0 = \sum_{i=1}^n (X_i - \bar{X})^2, \tilde{g}T_0 = c^2 T_0$. $T_0(gX) = c^2 T_0 = \tilde{g}T_0(X) \Rightarrow T_0$ is scale invariant. By Basu's theorem, T_0 is independent of Z. By Cor 4.1

$$T^*(X) = \frac{T_0(X)E_1[T_0(X)|Z]}{E_1[T_0^2(X)|Z]} = \frac{T_0(X)E_1[T_0(X)]}{E_1[T_0^2(X)]} = \frac{1}{n+1}\sum_{i=1}^n (X_i - \bar{X})^2.$$

In example 4.14, $T^*(X) = \dfrac{1}{n+2}\sum_{i=1}^n X_i$. Here μ is known. One more degree of freedom.

(d)

Let $T_0 = \sum_{i=1}^n (X_i - \bar{X})^2$. Stein's loss is scale invariant. T_0 is scale invariant.
MRIE $T^*(X) = \dfrac{T_0(X)}{u^*(Z)}$.

$$\begin{aligned} u^*(Z) &= \arg\min E_1[L(T_0(X)|u(Z))|Z = z] \\ &= \arg\min E_1\left[\frac{T_0}{u} - \log\frac{T_0}{u} - 1\right] \\ &= \arg\min\left(\frac{E_1(T_0)}{u} - E_1(\log T_0) + \log u - 1\right) \end{aligned}$$

Take derivative we get $u^*(Z) = E_1(T_0(X)) \Rightarrow T^*(X) = \dfrac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$.