Homework 5

Problem 1

(#2.64)(i)

Note that

$$\begin{aligned} R_{T_0}(\theta) &= E(T_0 - \theta)^2 \\ &= \theta^2 P(\bar{X} < 0.5) + (1-\theta)^2 P(\bar{X} > 0.5) + (0.5-\theta)^2 P(\bar{X} = 0.5). \end{aligned}$$

When $n = 2k$,

$$P(\bar{X} < 0.5) = \sum_{j=0}^{k-1} \binom{2k}{j} \theta^j (1-\theta)^{2k-j},$$

$$P(\bar{X} > 0.5) = \sum_{j=k+1}^{2k} \binom{2k}{j} \theta^j (1-\theta)^{2k-j},$$

and

$$P(\bar{X} = 0.5) = \binom{2k}{k} \theta^k (1-\theta)^k.$$

When $n = 2k+1$,

$$P(\bar{X} < 0.5) = \sum_{j=0}^{k} \binom{2k+1}{j} \theta^j (1-\theta)^{2k+1-j},$$

$$P(\bar{X} > 0.5) = \sum_{j=k+1}^{2k+1} \binom{2k+1}{j} \theta^j (1-\theta)^{2k+1-j},$$

and $P(\bar{X} = 0.5) = 0$.

(ii)

A direct calculation shows that

$$\begin{aligned} R_{T_1}(\theta) &= E(T_1 - \theta)^2 \\ &= \frac{1}{2} E(\bar{X} - \theta)^2 + \frac{1}{2} E(T_0 - \theta)^2 \\ &= \frac{\theta(1-\theta)}{2n} + \frac{1}{2} R_{T_0}(\theta), \end{aligned}$$

where $R_{T_0}(\theta)$ is given in part (i), and

$$\begin{aligned}
R_{T_2}(\theta) &= E(T_2-\theta)^2 \\
&= E\left[\bar{X}(\bar{X}-\theta)^2 + \left(\frac{1}{2}-\theta\right)^2(1-\bar{X})\right] \\
&= E(\bar{X}-\theta)^3 + \theta E(\bar{X}-\theta)^2 + \left(\frac{1}{2}-\theta\right)^2(1-\theta) \\
&= \frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} E(X_i-\theta)(X_j-\theta)(X_k-\theta) + \frac{\theta^2(1-\theta)}{n} + \left(\frac{1}{2}-\theta\right)^2(1-\theta) \\
&= \frac{E(X_1-\theta)^3}{n^2} + \frac{\theta^2(1-\theta)}{n} + \left(\frac{1}{2}-\theta\right)^2(1-\theta) \\
&= \frac{\theta(1-\theta)^3-\theta^3(1-\theta)}{n^2} + \frac{\theta^2(1-\theta)}{n} + \left(\frac{1}{2}-\theta\right)^2(1-\theta),
\end{aligned}$$

where the forth equality follows from $E(\bar{X}-\theta)^2 = Var(\bar{X}) = \theta(1-\theta)/n$ and the fifth equality follows from the fact that $E(X_i-\theta)(X_j-\theta)(X_k-\theta) \neq 0$ iff $i=j=k$.

(#2.69)

Here $T = 1/2$ means the decision rule is random with probability 0.5 to be 0 or 1.

$$\begin{aligned}
R(\theta,T) &= E(L(\theta,T)) \\
&= L(\theta,0)P(n\bar{X} < 0.5n) + L(\theta,1)P(n\bar{X} > 0.5n) + \frac{1}{2}[L(\theta,0)+L(\theta,1)]P(n\bar{X} = 0.5n).
\end{aligned}$$

If H_0 is true, i.e. $\theta \le 0.5$, then $L(\theta,0)=0$ and $L(\theta,1)=C_1$. So

$$R(\theta,T) = C_1\big[P(n\bar{X} > 0.5n) + 0.5P(n\bar{X} = 0.5n)\big].$$

If H_1 is true, i.e. $\theta > 0.5$, then $L(\theta,0)=C_0$ and $L(\theta,1)=0$. So

$$R(\theta,T) = C_0\big[P(n\bar{X} < 0.5n) + 0.5P(n\bar{X} = 0.5n)\big].$$

Using the fact that $n\bar{X} \sim Bin(n,\theta)$, we have

$$R(\theta,T) = \begin{cases} C_1\big[\sum_{j=0.5n+1}^{n}\binom{n}{j}\theta^j(1-\theta)^{n-j} + 0.5\binom{n}{0.5n}\theta^{0.5n}(1-\theta)^{0.5n}\big], & \theta \le 0.5 \\ C_0\big[\sum_{j=1}^{0.5n}\binom{n}{j}\theta^j(1-\theta)^{n-j} + 0.5\binom{n}{0.5n}\theta^{0.5n}(1-\theta)^{0.5n}\big], & \theta > 0.5 \end{cases}$$

Problem 2(#2.73)

i

Note that

$$\begin{aligned}
R_{a\bar{X}+b} &= E(a\bar{X}+b-\mu)^2 \\
&= a^2Var(\bar{X}) + (a\mu+b-\mu)^2 \\
&\ge a^2Var(\bar{X}) \\
&= a^2R_{\bar{X}(P)} \\
&> R_{\bar{X}}(P)
\end{aligned}$$

when $a > 1$. Hence $\bar{X}$ is better than $a\bar{X}+b$ with $a > 1$.

ii

For $b \neq 0$,

$$R_{\bar{X}+b}(P) = E(\bar{X} + b - \mu)^2 = Var(\bar{X}) + b^2 > Var(\bar{X}) = R_{\bar{X}}(P).$$

Hence $\bar{X}$ is better than $\bar{X} + b$ with $b \neq 0$.

Problem 3 (#2.81)

a

$X_{ij} \overset{iid}{\sim} N(\mu, \sigma_a^2 + \sigma_e^2)$.
Since the sample mean and sample variance are independent,

$$\bar{X}_i \perp \Sigma_{j=1}^n (X_{ij} - \bar{X}_i)^2.$$

Due to between-group independence,

$$\Sigma_{j=1}^n (X_{ij} - \bar{X}_i)^2 \perp \bar{X}_{i'}, i' \neq i.$$

Then $\Sigma_{j=1}^n (X_{ij} - \bar{X}_i)^2 \perp (\bar{X}_i - \bar{X})^2$. So MSA$\perp$MSE.

b

$\bar{X}_i \sim N(\mu, \sigma_a^2 + \frac{\sigma_e^2}{n})$, thus

$$\Sigma_i (\bar{X}_i - \bar{X})^2 \sim (\sigma_a^2 + \frac{\sigma_e^2}{n}) \mathcal{X}_{m-1}^2.$$

On the other hand, $X_{ij} - \bar{X}_i = \varepsilon_{ij} - \bar{\varepsilon}_i$ implies that

$$\Sigma_i \Sigma_j (X_{ij} - \bar{X}_i)^2 \sim \sigma_e^2 \mathcal{X}_{m(n-1)}^2.$$

Since MSA$\perp$MSE,

$$\frac{MSA}{MSE} \sim \frac{n\sigma_a^2 + \sigma_e^2}{\sigma_e^2} F_{m-1, m(n-1)}$$

and

$$E\left(\frac{MSA}{MSE}\right) = (n\theta + 1)\frac{mn - m}{mn - m - 2}.$$

Solving

$$\frac{1}{n}\left((1 - \delta)(n\theta + 1)\frac{mn - m}{mn - m - 2} - 1\right) = \theta,$$

we obtain

$$\delta = \frac{2}{m(n-1)}.$$

c

Since $\frac{MSA}{MSE} = (n\theta + 1) F_{m-1, m(n-1)}$ depends only on θ, the risk of $\hat{\theta}(\delta)$ can be expressed by δ and θ.

d

$$\begin{aligned}R(\theta,\hat\theta(\delta)) &= E(\hat\theta(\delta)-\theta)^2\\ &= E(\hat\theta(\delta)^2+\theta^2-2\theta\hat\theta(\delta))\\ &= (\delta-1)^2\frac{1}{n^2}E\left(\frac{MSA}{MSE}\right)^2+(\delta-1)\left(\frac{2}{n^2}E\left(\frac{MSA}{MSE}\right)+\frac{2\theta}{n}E\left(\frac{MSA}{MSE}\right)\right)+\theta^2+\frac{2\theta}{n}+\frac{1}{n^2}\end{aligned}$$

$E[(\hat\theta(\delta)-\theta)^2]$ is a convex quadratic function of δ, thus we conclude the statement.

e

From (d), $R(\theta,\hat\theta(\delta))$ attains minimum at

$$\delta^* = 1-\frac{(mn-m-4)(m-1)}{(m+1)m(n-1)} = \frac{2(mn+m-2)}{(m+1)m(n-1)},$$

$\delta=\frac{2}{m(n+2)}\neq\delta^*$,$R(\theta,\hat\theta(\delta))>R(\theta,\hat\theta(\delta^*))$, so $\hat\theta(\delta)$ is inadmissible.

Problem 4 (#4.80)

The joint Lebesgue density of $X_1,\dots,X_n$ is

$$\theta^{-n}e^{-n\bar X/\theta}I_{(0,\infty)}(X_{(1)}),$$

where $X_{(1)}$ is the smallest order statistic. Let $T(X)=\bar X,\vartheta=-\theta^{-1}$ and $c(\vartheta)=\vartheta^n$. Then the joint density is of the form $c(\vartheta)e^{\vartheta}T$ with respect a σ-finite measure and the range of ϑ is $(-\infty,0)$. For any $\vartheta_0\in(-\infty,0)$,

$$\int_{-\infty}^{\vartheta_0} e^{-b\vartheta/n}\vartheta^{-1}d\vartheta = \int_{theta_0}^{0} e^{-b\vartheta/n}\vartheta^{-1}d\vartheta=\infty.$$

By Karlin's theorem, we conclude that $(n\bar X+b)/(n+1)$ is admissible under the squared error loss. This implies that $(n\bar X+b)/(n+1)$ is also admissible under the loss function $L(\theta,a)=(a-\theta)^2/\theta^2$. Since the risk of $n\bar X/(n+1)$ is

$$\frac{1}{\theta}^2E\left(\frac{n\bar X}{n+1}-\theta\right)^2=\frac{1}{n+1},$$

$n\bar X/(n+1)$ is an admissible estimator with constant risk. Hence it is minimax.

Problem 5 (#4.75)

The posterior density of μ is proportional to

$$\exp\left\{-\frac{(\mu-x)^2}{2}+\mu\right\}\propto\exp\left\{-\frac{[\mu-(x+1)]^2}{2}\right\}.$$

Thus, the posterior distribution of μ is $N(X+1,1)$ and the generalized Bayes estimator is $E(\mu|X)=X+1$. Since the risk of $X+1$ is $E(X+1-\mu)^2=1+E(X-\mu)^2>E(X-\mu)^2$, which is the risk of X, we conclude that $X+1$ is neither minimax nor admissible.

Problem 6

(#4.81)

$$f(X_1,\ldots,X_n)=p^{n\bar{X}}(1-p)^{n-n\bar{X}}=(1-p)^n e^{n\bar{X}\log\frac{p}{1-p}}$$

Let $\theta=n\log\frac{p}{1-p}, c(\theta)=(1+e^{\frac{\theta}{n}})^{-n}, T(X)=\bar{X}, T_{\lambda,\gamma}(X)=(T+\gamma\lambda)/(1+\lambda)$. Let $\lambda=0$, we have that $T_{\lambda,\gamma}=T(X)$ and

$$\int_{\theta_0}^{\infty}1d\theta=\infty=\int_{-\infty}^{\theta_0}1d\theta,$$

so $T(X)=\bar{X}$ is admissible. $T_{\lambda,\gamma}(X)=(\bar{X}+\gamma\lambda)/(1+\lambda)$,

$$\begin{aligned}\int_{\theta_0}^{\infty}e^{-\gamma\lambda\theta}(1+e^{\frac{\theta}{n}})^{n\lambda}d\theta &= \int_{\theta_0}^{\infty}\exp\{-\gamma\lambda\theta+n\lambda\log(1+e^{\frac{\theta}{n}})\}d\theta \\ &\geq \int_{\theta_0}^{\infty}\exp\{\lambda\theta(1-\gamma)\}d\theta \qquad \textit{because} \quad \lambda>0 \\ &\geq \int_{\theta_0}^{\infty}\exp\{\lambda\theta_0(1-\gamma)\}d\theta=\infty \qquad \textit{because} \quad 0\leq 1-\gamma\leq 1\end{aligned}$$

So $(\bar{X}+\gamma\lambda)/(1+\lambda)$ is admissible for p.

(#4.82)(i)

The discrete probability density X is $\theta^x e^{-\theta}/x!=e^{-e^{\vartheta}}e^{\vartheta x}/x!$, where $\vartheta=\log\theta\in(-\infty,\infty)$. Let $\alpha=(1+\lambda)^{-1}$ and $\beta=\gamma\lambda/(1+\lambda)$.Since

$$\int_{-\infty}^{0}\frac{e^{-\gamma\lambda\vartheta}}{e^{-\lambda e^{\vartheta}}}d\vartheta=\infty$$

iff $\lambda\gamma\geq 0$, and

$$\int_{0}^{\infty}\frac{e^{-\gamma\lambda\vartheta}}{e^{-\lambda e^{\vartheta}}}d\vartheta=\infty$$

iff $\lambda\geq 0$. And by #2.73, when $\alpha=1$ and $\beta\neq 0$, $\alpha X+\beta$ is inadmissible. When $\alpha=0$ and $\beta=0$, $\alpha X+\beta$ is inadmissible. So the conclusion is that $\alpha X+\beta$ is admissible if and only if (α,β) is in the following set:

$$\{\alpha=0,\beta=0\}\cup\{\alpha=1,\beta=0\}\cup\{0<\alpha<1,\beta\geq 0\}.$$

(ii)

The discrete probability density of X is $\binom{x-1}{r-1}\frac{p^r}{(1-p)^r}e^{x\log(1-p)}$. Let $\theta=\log(1-p)\in(-\infty,0), \alpha=(1+\lambda)^{-1}$, and $\beta=\gamma\lambda/(1+\lambda)$. Note that

$$\int_{c}^{0}e^{-\lambda\gamma\theta}\left(\frac{e^{\theta}}{1-e^{\theta}}\right)^{\lambda r}d\theta=\infty$$

iff $\lambda r\geq 1$, i.e.,$\alpha\leq\dfrac{r}{r+1}$;

$$\int_{-\infty}^{c}e^{-\lambda\gamma\theta}\left(\frac{e^{\theta}}{1-e^{\theta}}\right)^{\lambda r}d\theta=\infty$$

iff $\gamma>r$, i.e.,$\beta>r\lambda/(1+\lambda)=r(1-\alpha)$. The result follows from Karlin's theorem.