Homework 6: Due 03/06/2018

1. (20 points) Consider squared error loss and construct James-Stein Estimator.

   (a) Why shrinkage estimators were more interesting than any other type of linear estimators? Let X be a r.v. in $\mathbb{R}$ with $E(X) = \mu$ and $Var(X) = \Sigma$. Consider linear estimators
   $$\delta_{a,b}(X) = aX + b, \quad \forall a \in \mathbb{R}, b \in \mathbb{R}^p.$$
   Show that $\delta_{ab}(X)$ is inadmissiable for either of the following three cases: (1) $a < 0$, or (2) $a > 1$, or (3) $a = 1, b \neq 0$.
   Remark: Letting $w = 1 - a$ and $\mu_0 = b/(1-a)$, this result suggests if we want an admissible linear estimator, it has to be in the form of shrinkage estimators:
   $$\delta(X) = (1-w)X + w\mu_0, \quad w \in [0,1], \mu_0 \in \mathbb{R}^p.$$

   (b) For simplicity, let's assume $b = 0, \Sigma = I_p$. Among all linear estimators $aX, a \in [0,1]$ (which are admissible), prove that $\hat{a}X$ has the smallest risk, where
   $$\hat{a} = \frac{\|\mu\|^2/p}{\|\mu\|^2/p + 1},$$
   and compare it with the risk of X. Remark: Obviously, $\hat{a} < 1$, hence it is an shrinkage estimator. But $\hat{a}X$ is not an estimator in the usual sense, since it contains unknown μ. (It is sometimes called "oracle estimator".) In the following, we will try to estimate it.

   (c) The form of $\hat{a}$ can be also derived from Baysian prespective. Consider the following hierarchical model:
   $$X = \mu + \varepsilon, \quad \varepsilon \sim N(0, I_p)$$
   $$\mu \sim N(0, \tau^2 I_p), \theta \perp\!\!\!\perp \varepsilon$$
   Prove that the Bayes estimator for μ is $\hat{\mu} = \tilde{a}X$ where
   $$\tilde{a} = \frac{\tau^2}{\tau^2 + 1}.$$
   Remark: in the previous part, $\|\mu\|^2/p$ can be viewed as the variance of μ, which is corresponding to τ^2 here.

   (d) Empirical Bayes method can be used to estimate τ^2 (or $\|\mu\|^2/p$). First show that the marginal distribution of X is $N(0, (\tau^2+1)I_p)$. Then, based on the sufficient and complete statistic $\|X\|^2$ for τ^2, derive the UMVUE for $\frac{\tau^2}{\tau^2+1}$. (Hint: $\|X\|^2 \sim Gamma$)
   Remark: Plug-in your UMVUE in the previous part, then you will get the James-Stein's estimator (with $c = 0$).

2. (30 points) Problem 96 +100 on page 312 of Shao (2003).