Homework 6

Problem 1

(a)

$$a > 1. \qquad R(\mu, \delta_{a,b}) = E[\sum_{i=1}^{p}(aX_i + b_i - \mu_i)^2]$$
$$= E[\sum_{i=1}^{p}(aX_i - a\mu_i + b_i - \mu_i + a\mu_i)^2]$$
$$= a^2E[\sum_{i=1}^{p}(X_i - \mu_i)^2] + \sum_{i=1}^{p}(b_i - \mu_i + a\mu_i)^2$$
$$\geq a^2R(\mu, X)$$
$$> R(\mu, X)$$

$$a = 1, b \neq 0 \qquad R(\mu, \delta_{a,b}) = E[\sum_{i=1}^{p}(X_i + b_i - \mu_i)^2]$$
$$= E(||x - \mu||^2) + \sum_{i=1}^{p} b_i^2$$
$$> R(\mu, X)$$

$$a < 0 \Rightarrow (a-1)^2 > 1 \qquad R(\mu, \delta_{a,b}) = E[\sum_{i=1}^{p}(aX_i - a\mu_i)^2 + \sum_{i=1}^{p}(b_i - \mu_i + a\mu_i)^2 + 0]$$
$$\geq \sum_{i=1}^{p}(b_i + \mu_i(a-1))^2$$
$$\geq (a-1)^2\sum_{i=1}^{p}(\mu_i - \frac{b_i}{a-1})^2$$
$$\geq \sum_{i=1}^{p}(\mu_i - \frac{b_i}{a-1})^2$$
$$= R(\mu, -\frac{b}{a-1})$$

So $\delta_{a,b}$ is inadmissible.

(b)

$$\begin{aligned}R(\mu, aX) &= \sum_{i=1}^{p} E(aX_i - \mu_i)^2 \\ &= a^2 \sum_{i=1}^{p}(\mu_i^2 + 1) + \sum_{i=1}^{p} \mu_i^2 - 2a \sum_{i=1}^{p} \mu_i^2 \\ &= a^2(||\mu||^2 + p) - 2a||\mu||^2 + ||\mu||^2\end{aligned}$$

When $\hat{a} = \dfrac{||\mu||^2/p}{||\mu||^2/p + 1}$, $R(\mu, \hat{a}X)$ attains minimum. So $\hat{a}X$ has the smallest risk.
$R(\mu, X) = \sum_{i=1}^{p} E(X_i - \mu)^2 = \sum_{i=1}^{p} Var(X_i) = p$
$R(\mu, \hat{a}X) = \dfrac{p}{1 + \frac{p}{||\mu||^2}} < p = R(\mu, X)$.

(c)

$$\begin{aligned}f(\mu|X) \propto f(X|\mu)f(\mu) &\propto \exp\{-\frac{1}{2}\sum_{i=1}^{p}(X_i - \mu_i)^2 - \frac{1}{2\tau^2}\sum_{i=1}^{p}\mu_i^2\} \\ &\propto \exp\{-\frac{1}{2}(1 + \frac{1}{\tau^2})\sum_{i=1}^{p}(\mu_i - (1 + \frac{1}{\tau^2})^{-1}X_i)^2\}\end{aligned}$$

$\Rightarrow \mu|X \sim N(\dfrac{\tau^2}{1+\tau^2}X, \dfrac{\tau^2}{1+\tau^2}I_p)$
$\Rightarrow \hat{\mu} = E(\mu|X) = \dfrac{\tau^2}{1+\tau^2}X$

(d)

$\varepsilon \perp \mu, X = \mu + \varepsilon$. Then $X \sim N(0, (1 + \tau^2)I_p)$. $\sum X_i^2 = ||X||^2$ is sufficient and complete for τ^2. If we find $\delta(X)$ with $E[\delta(X)] = \frac{\tau^2}{1+\tau^2}$, $E[\delta(X)|\sum X_i^2]$ is the UMVUE.
$\dfrac{\sum X_i^2}{1+\tau^2} \sim \mathcal{X}_p^2, E(\frac{1+\tau^2}{\sum X_i^2}) = \frac{1}{p-2}$ (inverse-chi square)

$$E(\frac{\sum X_i^2 - p + 2}{\sum X_i^2}) = 1 - E(\frac{p-2}{\sum X_i^2}) = 1 - \frac{1}{1+\tau^2} = \frac{\tau^2}{1+\tau^2}$$

So the UMVUE of $\frac{\tau^2}{1+\tau^2}$ is $\frac{\sum X_i^2 - p+2}{\sum X_i^2} = \frac{||x||^2 - p+2}{||x||^2}$

Problem 2

(#4.96)(a)

Let $X_{(n)}$ be the largest order statistic. The likelihood function is $l(\theta) = \theta^{-n} I_{\{X_{(n)},\ldots,\theta_0\}}$, which is 0 when $\theta < X_{(n)}$ and decreasing on $\{X_{(n)}, \ldots, \theta_0\}$. Hence, the MLE of θ is $X_{(n)}$.

(b)

Let $X_{(1)}$ be the smallest order statistic. The likelihood function is $l(\theta) = \exp\{-\sum_{i=1}^n (X_i - \theta)\} I_{(0,X_{(1)})}(\theta)$, which is 0 when $\theta > X_{(1)}$ and increasing on $(0, X_{(1)})$. Hence, the MLE of θ is $X_{(1)}$.

(c)

Note that $l(\theta) = \theta^n \Pi_{i=1}^n (1 - X_i)^{\theta-1} I_{(0,1)}(X_i)$ and, when $\theta > 1$,

$$\frac{\partial \log l(\theta)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^n \log(1 - X_i)$$

and

$$\frac{\partial^2 \log l(\theta)}{\partial \theta^2} = -\frac{n}{\theta^2} < 0.$$

The equation $\frac{\partial \log l(\theta)}{\partial \theta} = 0$ has a unique solution $\hat{\theta} = -n / \sum_{i=1}^n \log(1 - X_i)$. If $\hat{\theta} > 1$, then it maximizes $l(\theta)$. If $\hat{\theta} \leq 1$, then $l(\theta)$ is decreasing on the interval $(1, \infty)$. Hence the MLE of θ is $\max\{1, \hat{\theta}\}$.

(d)

Note that

$$\frac{\partial \log l(\theta)}{\partial \theta} = \frac{n}{\theta(1-\theta)} + \frac{1}{(1-\theta)^2} \sum_{i=1}^n \log X_i$$

and $\frac{\partial \log l(\theta)}{\partial \theta} = 0$ has a unique solution $\hat{\theta} = (1 - n^{-1} \sum_{i=1}^n \log X_i{}^{-1}) < 1$. Also, $\frac{\partial \log l(\theta)}{\partial \theta} < 0$ when $\theta > \hat{\theta}$ and $\frac{\partial \log l(\theta)}{\partial \theta} > 0$ when $\theta < \hat{\theta}$. Hence, the MLE of θ is $\max\{\hat{\theta}, \frac{1}{2}\}$.

(e)

Note that $l(\theta) = 2^{-n} \exp\{-\sum_{i=1}^n |X_i - \theta|\}$. Let F_n be the distribution putting mass n^{-1} to each X_i. Then any median of F_n is an MLE of θ.

(f)

Since $l(\theta) = \theta^n \Pi_{i=1}^n X_i^{-2} I_{(0,X_{(1)})}(\theta)$, the same argument in part (b) of the solution yields the MLE $X_{(1)}$.

(g)

Let $\bar{X}$ be the sample mean. Since

$$\frac{\partial \log l(\theta)}{\partial \theta} = \frac{n\bar{X}}{\theta} - \frac{n - n\bar{X}}{1 - \theta},$$

$\frac{\partial \log l(\theta)}{\partial \theta} = 0$ has a unique solution $\bar{X}$, $\frac{\partial \log l(\theta)}{\partial \theta} < 0$ when $\theta > \bar{X}$, and $\frac{\partial \log l(\theta)}{\partial \theta} > 0$ when $\theta < \bar{X}$. Hence, the same argument in part (d) yields the MLE

$$\hat{\theta} = \begin{cases} \frac{1}{2} & \text{if } \bar{X} \in [0, \frac{1}{2}) \\ \bar{X} & \text{if } \bar{X} \in [\frac{1}{2}, \frac{3}{4}) \\ \frac{3}{4} & \text{if } \bar{X} \in (\frac{3}{4}, 1]. \end{cases}$$

(h)

Note that

$$\log l(\theta) = -\frac{n}{2}\log(2\pi\theta^2) - \sum_{i=1}^{n}\frac{(X_i-\theta)^2}{2\theta^2}$$

$$\frac{\partial \log l(\theta)}{\partial\theta} = -\frac{1}{\theta^3}\left(n\theta^2 + \theta\sum_{i=1}^{n}X_i - \sum_{i=1}^{n}X_i^2\right).$$

The equation $\frac{\partial \log l(\theta)}{\partial\theta} = 0$ has two solutions

$$\theta_{\pm} = \frac{-\sum_{i=1}^{n}X_i \pm \sqrt{(\sum_{i=1}^{n}X_i)^2 + 4n\sum_{i=1}^{n}X_i^2}}{2n}.$$

Note that $\lim_{\theta\to 0}\log l(\theta) = \lim_{\theta\to\pm\infty}\log l(\theta) = -\infty$. We conclude that both θ_- and θ_+ are local maximum points. Therefore, the MLE of θ is

$$\hat{\theta} = \begin{cases}\theta_- & \text{if } l(\theta_-) \geq l(\theta_+)\\ \theta_+ & \text{if } l(\theta_-) < l(\theta_+).\end{cases}$$

(i)

The likelihood function

$$l(\theta) = \sigma^{-n}\exp\left\{-\frac{1}{\sigma}\sum_{i=1}^{n}(X_i-\mu)\right\}I_{(0,X_{(1)})}(\mu)$$

is 0 when $\mu > X_{(1)}$ and increasing on $(0, X_{(1)})$. Hence the MLE of μ is $X_{(1)}$. Substituting $\mu = X_{(1)}$ into $l(\theta)$ and maximizing the resulting likelihood function yields that the MLE of σ is $n^{-1}\sum_{i=1}^{n}(X_i - X_{(1)})$.

(j)

Let $Y_i = \log X_i$, $i = 1,\ldots,n$. Then

$$l(\theta) = \frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i-\mu)^2 - \sum_{i=1}^{n}Y_i\right\}.$$

Solving $\frac{\partial \log l(\theta)}{\partial\theta} = 0$, we obtain the MLE of μ as $\bar{Y} = n^{-1}\sum_{i=1}^{n}Y_i$ and the MLE of σ^2 as $n^{-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$.

(k)

Since $l(0) = 1$ and $l(1) = (2^n\Pi_{i=1}^{n}\sqrt{X_i})^{-1}$, the MLE is equal to 0 if $2^n\Pi_{i=1}^{n}\sqrt{X_i} < 1$ and is equal to 1 if $2^n\Pi_{i=1}^{n}\sqrt{X_i} \geq 1$.

(l)

The likelihood function is $l(\theta) = \alpha^n\beta^{-n\alpha}\Pi_{i=1}^{n}X_i^{\alpha-1}I_{(X_{(n)},\infty)}(\beta)$, which is 0 when $\beta < X_{(n)}$ and decreasing in β otherwise. Hence the MLE of β is $X_{(n)}$. Substituting $\beta = X_{(n)}$ into the likelihood function, we obtain the MLE of α as $n[\sum_{i=1}^{n}\log(X_{(n)}/X_i)]^{-1}$.

(m)

Let $X_{(n)}$ be the largest X_i's and $T = \sum_{i=1}^n X_i$. Then

$$l(\theta) = \prod_{i=1}^n \binom{\theta}{X_i} p^T (1-p)^{n\theta - T} I_{\{X_{(n)}, X_{(n)}+1, \dots\}}(\theta).$$

For $\theta = X_{(n)}, X_{(n)} + 1, \dots,$

$$\frac{l(\theta+1)}{l(\theta)} = (1-p)^n \prod_{i=1}^n \frac{\theta+1}{\theta+1-X_i}.$$

Since $(\theta+1)/(\theta+1-X_i)$ is decreasing in θ, the function $l(\theta+1)/l(\theta)$ is decreasing in θ. Also, $\lim_{\theta\to\infty} l(\theta+1)/l(\theta) = (1-p)^n < 1$. Therefore, the MLE of θ is $\max\{\theta : \theta \geq X_{(n)}, l(\theta+1)/l(\theta) \geq 1\}$.

(n)

Let $\bar{X}$ be the sample mean. Then

$$l(\theta) = \frac{1}{2^n}(1-\theta^2)^n \exp\{n\theta\bar{X} - |\bar{X}|\}$$

$$\frac{\partial \log l(\theta)}{\partial\theta} = n\bar{X} - \frac{2n\theta}{1-\theta^2}$$

. When $\bar{X} \neq 0$, the equation $\frac{\partial \log l(\theta)}{\partial\theta} = 0$ has two solutions $\theta_\pm = \frac{\pm\sqrt{1+\bar{X}^2}-1}{\bar{X}}$. Since $\theta_- \leq -1$ is not in the parameter space, we conclude that the MLE of θ is $\frac{\sqrt{1+\bar{X}^2}-1}{\bar{X}}$. When $\bar{X} = 0$, the MLE of θ is 0.

(# 4.100)(a)

Let $l(\lambda, \mu)$ be the likelihood function, $\bar{Y} = n^{-1}\sum_{i=1}^n Y_i$, and $\bar{Z} = n^{-1}\sum_{i=1}^n Z_i$. Since Y_i's and Z_i's are independent,

$$\frac{\partial \log l(\lambda,\mu)}{\partial\lambda} = -\frac{n}{\lambda} + \frac{n\bar{Y}}{\lambda^2} \text{ and } \frac{\partial \log l(\lambda,\mu)}{\partial\mu} = -\frac{n}{\mu} + \frac{n\bar{Z}}{\mu^2}.$$

Hence, the MLE of (λ, μ) is $(\bar{Y}, \bar{Z})$.

(b)

The probability density of (X_i, Δ_i) is $\lambda^{-\Delta_i}\mu^{-(\Delta_i - 1)}e^{-(\lambda^{-1}+\mu^{-1})x_i}$. Let $T = \sum_{i=1}^n X_i$ and $D = \sum_{i=1}^n \Delta_i$. Then

$$l(\lambda,\mu) = \lambda^{-D}\mu^{D-n}e^{-(\lambda^{-1}+\mu^{-1})T}.$$

If $0 < D < n$, then

$$\frac{\partial \log l(\lambda,\mu)}{\partial\lambda} = -\frac{D}{\lambda} + \frac{T}{\lambda^2} \text{ and } \frac{\partial \log l(\lambda,\mu)}{\partial\mu} = \frac{D-n}{\mu} + \frac{T}{\mu^2}.$$

The likelihood equation has a unique solution $\hat{\lambda} = T/D$ and $\hat{\mu} = T/(n - D)$. The MLE of (λ, μ) is $(\hat{\lambda}, \hat{\mu})$.
If $D = 0$,

$$l(\lambda, \mu) = \mu^{-n} e^{-(\lambda^{-1} + \mu^{-1})T},$$

which is increasing in λ. Hence there does not exist an MLE of λ. Similarly, when $D = n$, there does not exist an MLE of μ.