# Homework 7: Due 04/03/2018

1. (5 points) Problem 105 on page 313 of Shao (2003).

2. (10 points) Problem 112 on page 314 of Shao (2003).

3. (15 points) Consider a measurement noise model

$$X_i = \theta + e_i, \quad i = 1, 2, \cdots, n; \theta \in \mathbb{R},$$

where $e_i$ is the measurement error iid with $E(e_i) = 0$. To estimate $\theta$, consider the sample mean and sample median of $X_1, \cdots, X_n$. Discuss the asymptotic behavior of the two estimators in the following cases.

   (a) The $e_i$ follows the standard Laplace distribution with pdf

$$f_0(t) = \frac{1}{2} e^{-|t|}, \quad t \in \mathbb{R}.$$

   (b) The $e_i$ follows the standard Normal distribution with pdf

$$f_0(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \quad t \in \mathbb{R}.$$

   (c) The $e_i$ follows the standard Logistic distribution with pdf

$$f_0(t) = \frac{e^{-t}}{(1+e^{-t})^2}, \quad t \in \mathbb{R}.$$

   (d) The $e_i$ follows student's-t distribution with $\nu$ degree of freedom, for $\nu = 3, 4, 5$.

Hint: You can use the fact from Chapter 5 (Theorem 5.10) about the asymptotic distribution of sample median,

$$\sqrt{n}(\hat{\theta}_{1n} - \theta) \xrightarrow{D} N(0, [2f(\theta)]^{-2}).$$

where $f$ is the cdf of $X_i$ ($f(\theta) = f_0(0)$).

4. (15 points) Let $X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\mu, 1), \mu \in \mathbb{R}$. Let $\theta = P(X_1 \leq c)$ be the parameter of interest for some fixed constant c.

   (a) Find the MLE, $\hat{\theta}_{1n}$, of $\theta$, and derive the asymptotic distribution.

   (b) Find the UMVUE, $\hat{\theta}_{2n}$, of $\theta$, and compare with $\hat{\theta}_{1n}$.

   (c) Without using normality, one may derive the following nonparametric estimator,

$$\hat{\theta}_{3n} = F_n(c) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(X_i \leq c).$$

   Derive the asymptotic distribution of $\hat{\theta}_{3n}$, and compare with the above estimators.

(d) Consider the Bayes estimator with the prior $\mu \sim N(\mu_0, \sigma_0^2)$, $\hat{\theta}_{4n}$, of $\theta$. Compare with the above estimators.

5. (5 points) Suppperefficiency. Let $X_1, \cdots, X_n \overset{\text{iid}}{\sim} N(\theta, 1), \theta \in \mathbb{R}$ and consider a version of Hodge estimator

$$\hat{\theta}_n = \bar{X}\mathbb{1}\{\bar{X} > n^{-1/4}\} = \begin{cases} \bar{X}, & \text{if } \bar{X} > n^{-1/4} \\ 0, & \text{otherwise} \end{cases}$$

This estimator shrinks to 0 for a very small $\bar{x}$. Show the asymptotic distribution of $\hat{\theta}_n$,

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{D} N(0, v), \quad \text{where } v = \mathbb{1}\{\theta = 0\}.$$