Homework 7

Problem 1 (# 4.105)

The log-likelihood function is

$$\log l(\alpha, \theta) = n \log \alpha - n \log \theta + (\alpha - 1) \sum_{i=1}^{n} \log X_i - \frac{1}{\theta} \sum_{i=1}^{n} X_i^{\alpha}.$$

Hence, the likelihood equations are

$$\frac{\partial \log l(\alpha, \theta)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log X_i - \frac{1}{\theta} \sum_{i=1}^{n} X_i^{\alpha} \log X_i = 0$$

and

$$\frac{\partial \log l(\alpha, \theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} X_i^{\alpha} = 0,$$

which are equivalent to $h(\alpha) = n^{-1} \sum_{i=1}^{n} \log X_i$ and $\theta = n^{-1} \sum_{i=1}^{n} X_i^{\alpha}$. Note that

$$h'(\alpha) = \frac{\sum_{i=1}^{n} X_i^{\alpha} (\log X_i)^2 \sum_{i=1}^{n} X_i^{\alpha} - (\sum_{i=1}^{n} X_i^{\alpha} \log X_i)^2}{(\sum_{i=1}^{n} X_i^{\alpha})^2} + \frac{1}{\alpha^2} > 0$$

by the Cauchy-Schwarz inequality. Thus, $h(\alpha)$ is increasing. Since h is continuous, $\lim_{\alpha \to 0} h(\alpha) = -\infty$, and

$$\lim_{\alpha \to \infty} h(\alpha) = \lim_{\alpha \to \infty} \frac{\sum_{i=1}^{n} (\frac{X_i}{X_{(n)}})^{\alpha} \log X_i}{\sum_{i=1}^{n} (\frac{X_i}{X_{(n)}})^{\alpha}} = \log X_{(n)} > \frac{1}{n} \sum_{i=1}^{n} \log X_i,$$

where $X_{(n)}$ is the largest order statistic and the inequality holds as long as X_i's are not identical, we conclude that the likelihood equations have a unique solution.

Problem 1 (# 4.112)

i

Let $X_{(n)}$ be the largest order statistic. Then $\hat{\theta} = X_{(n)}$ and $T(X) = \frac{n+1}{n} X_{(n)}$. The mean squared error of $\hat{\theta}$ is

$$E(X_{(n)} - \theta)^2 = \frac{2\theta^2}{(n+1)(n+2)}$$

and the mean squared error of T is

$$E(T - \theta)^2 = \frac{\theta^2}{n(n+2)}.$$

The ratio is $\frac{n+1}{2n}$. When $n \geq 2$, this ratio is less than 1 and, therefore, the MLE $\hat{\theta}$ is inadmissible.

ii

From

$$\begin{aligned}P(n(\theta-\hat{\theta})\leq x) &= P(X_{(n)}\geq \theta-\frac{x}{n})\\ &= \theta^{-n}\int_{\theta-x/n}^{\theta} nt^{n-t}dt\\ &= 1-(1-\frac{x}{n\theta})^n\\ &\to 1-e^{-x/\theta}\end{aligned}$$

as $n\to\infty$, we conclude that $n(\theta-\hat{\theta})\to_d Z_{0,\theta}$. From

$$n(\theta-T)=n(\theta-\hat{\theta})-\hat{\theta}$$

and Slutsky's theorem, we conclude that $n(\theta-T)\to_d Z_{0,\theta}-\theta$, which has the same distribution as $Z_{-\theta,\theta}$. The asymptotic relative efficiency of $\hat{\theta}$ with respect to T is $E(Z^2_{-\theta,\theta})/E(Z^2_{0,\theta})=\theta^2/(\theta^2+\theta^2)=\frac{1}{2}$.

Problem 3

(a)

Denote the sample median as $\hat{\theta}_{1n}$. $\hat{\theta}_{1n}$ is the MLE.

$$\sqrt{n}(\hat{\theta}_{1n}-\theta)\to_D N(0,I^{-1}(\theta_0))$$

where $I(\theta_0)=var(sign(X_i-\theta))=E(sign(X_i-\theta)^2)=1$. By CLT: $\sqrt{n}(\bar{X}_n-\theta)\to_D N(0,2)$ since $var(X_i)=var(e_i)=2$, $are(\hat{\theta}_{1n},\bar{X})=\frac{2}{1}=2$. So $\hat{\theta}_{1n}$ is better.

(b)

By CLT: $\sqrt{n}(\bar{X}_n-\theta)\to_D N(0,1)$. Since $var(X_i)=var(e_i)=1$, $f(\theta)=f_0(0)=(2\pi)^{-\frac{1}{2}}$. By theorem: $\sqrt{n}(\hat{\theta}_{1n}-\theta)\to_D N(0,[2f(\theta)]^{-2})=N(0,\frac{\pi}{2})$. $are(\hat{\theta}_{1n},\bar{X})=\frac{1}{\pi/2}=\frac{2}{\pi}<1$. So $\bar{X}$ is better.

(c)

$\sqrt{n}(\bar{x}_n-\theta)\to_D N(0,\frac{\pi^2}{3})$ because the variance of the logistic distribution is $\frac{\sigma^2\pi^2}{3}$. $\sqrt{n}(\hat{\theta}_{1n}-\theta)\to_D N(0,[2f(\theta)]^{-2})$, $f(\theta)=f_0(0)=\frac{1}{4}$, $are(\hat{\theta}_{1n},\bar{X})=\frac{\pi^2}{12}<1$. $\bar{X}$ is better.

(d)

$var(X_i)=var(e_i)=\frac{\gamma}{\gamma-2}$, $f(0)=f_0(0)=\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\gamma\pi}\Gamma\left(\frac{\gamma}{2}\right)}$. $\sqrt{n}(\bar{X}-\theta)\to_D N(0,\frac{\gamma}{\gamma-2})$, $\sqrt{n}(\hat{\theta}_{1n}-\theta)\to_D N\left(0,\left(\frac{2\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\gamma\pi}\Gamma\left(\frac{\gamma}{2}\right)}\right)^{-2}\right)$.

$$are(\hat{\theta}_{1n},\bar{X}) = \frac{\frac{\gamma}{\gamma-2}}{\left(\frac{2\Gamma(\frac{\gamma+1}{2})}{\sqrt{\gamma\pi}\Gamma(\frac{\gamma}{2})}\right)^{-2}} = \begin{cases} \frac{16}{\pi^2} > 1 & \gamma = 3 \\ \frac{9}{8} > 1 & \gamma = 4 \\ \frac{256}{27\pi^2} < 1 & \gamma = 5 \end{cases} \tag{1}$$

$\hat{\theta}_{1n}$ is better when $\gamma = 3, 4$; $\bar{X}$ is better when $\gamma = 5$.

Problem 4

(a)

$\bar{X}$ is MLE of μ. $\hat{\theta}_{1n} = \Phi(c - \hat{\mu}) = \Phi(c - \bar{X})$ is the MLE of θ. By CLT, $\sqrt{n}(\hat{X} - \mu) \to_D N(0, 1)$. By δ-method, $\sqrt{n}(\Phi(c - \bar{X}) - \Phi(c - \mu)) \to_D N[0, (\Phi'(c - \mu))^2] = N(0, \frac{1}{2\pi}e^{-(c-\mu)^2})$. So $\sqrt{n}(\hat{\theta}_{1n} - \theta) \to_D N(0, \frac{1}{2\pi}e^{-(c-\mu)^2})$.

(b)

$E[1(X_1 \le c)] = P(X_1 \le c) = \theta$, $\bar{X}$ is complete and sufficient. By Rao-Blackwell's thm, $E[1(X_1 \le c)|\bar{X}] = P(X_1 \le c|\bar{X})$ is the UMVUE of $P(X_1 \le c)$. $(X_1, \bar{X}) \sim N_2\left(\mu, \begin{pmatrix} 1 & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} \end{pmatrix}\right) \Rightarrow$ $X_1|\bar{X} \sim N(\bar{X}, 1 - \frac{1}{n}) \Rightarrow \hat{\theta}_{2n} = P(X_1 \le c|\bar{X}) = \Phi\left(\frac{c - \bar{X}}{\sqrt{1 - \frac{1}{n}}}\right)$ is the UMVUE of θ. By δ-method $\sqrt{n}\left(\Phi\left(\frac{c - \bar{X}}{\sqrt{1 - \frac{1}{n}}}\right) - \Phi\left(\frac{c - \mu}{\sqrt{1 - \frac{1}{n}}}\right)\right) \to_D N\left(0, \frac{1}{2\pi}e^{-\frac{(c-\mu)^2}{(1-\frac{1}{n})}}\right)$. So $are(\hat{\theta}_{1n} - \hat{\theta}_{2n}) = \lim_{n\to\infty} \frac{var(\hat{\theta}_{2n})}{var(\hat{\theta}_{1n})} = \lim_{n\to\infty} \frac{e^{-\frac{(c-\mu)^2}{(1-\frac{1}{n})}}}{e^{-(c-\mu)^2}} = 1$. So $\hat{\theta}_{1n}$ and $\hat{\theta}_{2n}$ are asymptotically equivalent.

(c)

$1(X_i \le c) \sim Bin(\theta, 1)$, $\theta = P(X_i \le c) = \Phi(c - \mu)$. By CLT,

$$\sqrt{n}(\hat{\theta}_{3n} - \theta) \to_D N(0, \theta(1 - \theta)),$$

$$are(\hat{\theta}_{1n}, \hat{\theta}_{3n}) = \frac{\theta(1-\theta)}{\phi^2(c-\mu)} = \frac{\Phi(c-\mu)[1 - \Phi(c-\mu)]}{\phi^2(c-\mu)}.$$

Let $t = c - \mu$, then

$$\frac{d}{dt}are(\hat{\theta}_{1n}, \hat{\theta}_{3n}) = \frac{d}{dt}\frac{\Phi(t)[1 - \Phi(t)]}{\phi^2(t)} = \frac{1 - 2\Phi(t)}{\phi(t)},$$

which is < 0 for $t < 0$, > 0 for $t > 0$, and $= 0$ iff $t = 0$. Therefore, the $are(\hat{\theta}_{1n}, \hat{\theta}_{3n})$ is maximized at $c = \mu$ and

$$\max are(\hat{\theta}_{1n}, \hat{\theta}_{3n}) = \sqrt{2\pi}/4 \approx 0.627 < 1,$$

which implies the MLE of θ is asymptotically more efficient than the nonparametric estimator $\hat{\theta}_{3n}$.

(d)

First $\mu|X \sim N(\mu^*, d^2)$, where

$$\mu_* = \frac{\sigma^2}{n\sigma_0^2 + \sigma^2}\mu_0 + \frac{n\sigma_0^2}{n\sigma_0^2 + \sigma^2}\bar{X} \quad \text{and} \quad d^2 = \frac{\sigma^2\sigma_0^2}{n\sigma_0^2 + \sigma^2}.$$

Furthermore, with respect to squared error loss function, $\hat{\theta}_{4n} = E[\Phi(c-\mu)|X]$.
Here $\Phi(c-\mu)|X$ is a nonlinear transformation of $\mu|X$, whose mean is hard to evaluate analytically. But we can consider the conditional mean given $\bar{X}$ in the asymptotic sense. Since

$$\mu - \bar{X}|\bar{X} \sim N\big(\frac{\sigma^2}{n\sigma_0^2 + \sigma^2}(\bar{X} - \mu_0), \ \ \frac{\sigma^2\sigma_0^2}{n\sigma_0^2 + \sigma^2}\big),$$

we have

$$\sqrt{n}(\mu - \bar{X})|\bar{X} \to_D N(0, \sigma^2).$$

By delta-method,

$$\sqrt{n}(\Phi(c-\mu) - \Phi(c-\bar{X})) \mid \bar{X} \to_D N(0, \phi^2(c-\bar{X})\sigma^2) = N(0, \phi^2(c-\bar{X})).$$

That is

$$\Phi(c-\mu)|\bar{X} = \Phi(c-\bar{X}) + \frac{1}{\sqrt{n}}N(0, \phi^2(c-\bar{X})) + o_P(n^{-1/2}).$$

Hence $\hat{\theta}_{4n} = E(\Phi(c-\mu)|\bar{X}) = \Phi(c-\bar{X})) + o_P(n^{-1/2})$. Therefore, the Bayesian estimator is asymptotically equivalent to the MLE up the order $n^{-1/2}$.

Problem 5

When $\theta \neq 0$

$$\begin{aligned} P(|\bar{X}| \leq n^{-\frac{1}{4}}) &= P(\sqrt{n}(-n^{-\frac{1}{4}} - \theta) \leq \sqrt{n}(\bar{X} - \theta) \leq \sqrt{n}(n^{-\frac{1}{4}} - \theta)) \\ &= \Phi(\sqrt{n}(n^{-\frac{1}{4}} - \theta)) - \Phi(\sqrt{n}(-n^{-\frac{1}{4}} - \theta)) \\ &\to 0 \qquad\qquad \text{as } n \to \infty. \end{aligned}$$

So $P(\theta_n \hat{=} \bar{X}) \to 1$. So $\sqrt{n}(\hat{\theta}_n - \theta) = \sqrt{n}(\hat{\theta} - \bar{X}_n) + \sqrt{n}(\bar{X}_n - \theta) \to_D N(0,1)$ by CLT.
When $\theta = 0$, $\bar{X} \to_{a.s.} 0$. So $P(\hat{\theta} = 0) \to 1$. Thus $\sqrt{n}(\theta_n - \theta) \to_D N(0,0)$.