# Homework 9: Due 04/17/2018

1. (35 points) Consider the kernel density estimation discussed in class. Relax the assumption of the symmetry for the kernel function. Let $n \to \infty$ and $h_n \asymp n^{-\alpha}$, for some $\alpha > 0$.

   (a) Suppose $f \in C^1$. For any fixed t, show that the leading term of MSE is

   $$MSE(\hat{f}_n) = h_n^2 \mu_K^2 (f'(t))^2 + o(h_n^2) + \frac{1}{nh_n} f(t)\|K\|^2 + o(\frac{1}{nh_n}) + O(\frac{1}{n}),$$

   where $\mu_k = \int uK(u)du$.

   (b) Let $n \to \infty$. Derive the order of $h_n$ to minimize above MSE. That is, find $\alpha$ to minimize the MSE.

   (c) Based on the optimal $\alpha$ in the part (b), what are the orders of $MSE(\hat{f}_n)$, $E(\hat{f}_n)$, and $Var(\hat{f}_n)$?

   (d) Let $nh_n^3 \to \infty$, $h_n \to 0$, and $n \to \infty$. Show that

   $$\sqrt{nh_n}(\hat{f}_n(t) - f(t)) \xrightarrow{D} N(0, \|K\|^2 f(t)).$$

   Remark: Note here $h_n = o(n^{-1/3})$. Comparing result in part (c) and (d), one can choose a slightly smaller $h_n$ to kill the bias and get a slightly larger variance. So the convergence rate in part (d) is slightly worse than part (c), but without the first order asymptotic bias.

   (e) Suppose $f$ is bounded and continuous on $[a, b]$. Show that

   $$\int_a^b \hat{f}_n(t)dt \xrightarrow{p} \int_a^b f dt.$$

2. (15 points) Using DKW inequality for empirical cdf and Borel-Cantelli lemma, show the strong uniform consistency of $\hat{f}_n$ with uniform kernel. That is, if $n \to \infty$ and $nh_n^2 / \log(n) \to 0$, then

   $$\sup_{t \in \mathbb{R}} |\hat{f}_n - f(t)| \xrightarrow{a.s} 0.$$