Homework 9

Problem 1

a

Bias $(\hat{f}_n(t)) = E(\hat{f}_n(t)) - f(t) = E(\frac{1}{nh_n}\sum_{i=1}^n K(\frac{t-X_i}{h_n})) - f(t) = E(\frac{1}{h_n}\sum_{i=1}^n K(\frac{t-X_1}{h_n})) - f(t)$, Let $u = \frac{t-X_1}{h_n}$, we get the bias is equal to

$$\int_{-\infty}^{\infty} K(u)[f(t-uh_n) - f(t)]du.$$

By Taylor's expansion,

$$f(t) = f(t-uh_n) + uh_n f^{'}(t) + o(h_n)$$

So bias

$$(\hat{f}_n(t)) = -h_n f^{'}(t)\int_{-\infty}^{\infty} K(u)udu + o(h_n) = -h_n f^{'}(t)\mu_k + o(h_n);$$

$$Var(\hat{f}_n(t)) = \frac{1}{n}\{E[\frac{1}{h_n^2}k^2(\frac{t-X_1}{h_n})] - [E(\frac{1}{h_n}K(\frac{t-X_1}{h_n}))]^2\}$$

$$= \frac{1}{n}\{\int_{-\infty}^{\infty}\frac{1}{h_n}K^2(u)[f(t)-uh_n]du - [\int_{-\infty}^{\infty}K(u)f(t-uh_n)du]^2\}$$

$$= \frac{1}{n}\{\int_{-\infty}^{\infty}\frac{1}{h_n}K^2(u)[f(t)-o(1)]du - O(1)\}$$

$$= \frac{1}{n}\{\frac{1}{h_n}f(t)||K||^2 + o(\frac{1}{h_n}) - O(1)\}$$

$$= \frac{1}{nh_{n]}}f(t)||K||^2 + o(\frac{1}{nh_n}) + O(\frac{1}{n})$$

Thus, mean square error:

$$Bias(\hat{f}_n)^2 + var(\hat{f}_n) = {h_n}^2{\mu_k}^2(f^{'}(t))^2 + o({h_n}^2) + \frac{1}{nh_n}f(t)||K||^2 + o(\frac{1}{nh_n}) + O(\frac{1}{n})$$

b

Taking derivative of MSE w.r.t. h_n and set it to 0.

$$C_1 h_n - \frac{c_2}{nh_n^2} = 0$$

C_1 and C_2 are constant and $h_n \asymp (\frac{1}{n})^{1/3}$, so $\alpha = \frac{1}{3}$

c

$MSE(\hat{f}_n) \asymp n^{-2/3}$, $Var(\hat{f}_n) \asymp n^{-2/3}$ and

$$E(\hat{f}_n) = f(t) + Bias(\hat{f}_n) = -h_n\mu_k f^{'}(t) + o(h_n) + f(t) \asymp n^{-1/3}$$

d

$h_n = o(n^{-\frac{1}{3}})$ is equivalent to $h_n^3 \to 0$

$$\hat{f}_n(t) - f(t) = (\hat{f}_n(t) - E(\hat{f}_n(t))) + Bias(\hat{f}_n)$$

By (a), $Bias(\hat{f}_n) = O(h_n)$.

$$\Rightarrow lim_{n\to\infty}\sqrt{nh_n}(Bias(\hat{f}_n)) = lim_{n\to\infty}O(\sqrt{nh_n^3}) = 0$$

By (a)

$$E[\sqrt{nh_n}(\hat{f}_n - E[\hat{f}_n(t)])]^2 = f(t)||K||^2 + nh_n o(\frac{1}{nh_n}) + nh_n O(\frac{1}{n})$$

$$\to f(t)||K||^2 asn \to \infty$$

By CLT we know that

$$\sqrt{nh_n}(f_n(t)\hat{-} f(t)) \xrightarrow{D} N(0, f(t)||K||^2)$$

e

$lim_{n\to\infty}E\int_a^b \hat{f}_n(t)dt = lim_{n\to\infty}\int_a^b E[\hat{f}_n(t)]dt$

$$= lim_{n\to\infty}\int_a^b\int_{-\infty}^{\infty} K(u)f(t-h_nu)dudt$$

$$= \int_a^b\int_{-\infty}^{\infty} K(u)f(t)dudt = \int_a^b f(t)dt$$

For $s \neq t$,

$$E[\hat{f}_n(s)\hat{f}_n(t)] = \frac{1}{n^2h_n^2}E[\sum_{i=1}^n\sum_{j=1}^n K(\frac{t-X_i}{h_n})K(\frac{t-X_j}{h_n})]$$

$$= \frac{1}{nh_n^2}E[K(\frac{t-X_1}{h_n})K(\frac{s-X_1}{h_n})] + \frac{n-1}{nh_n^2}E[K(\frac{t-X_1}{h_n})]E[K(\frac{t-X_1}{h_n})]$$

$$= \frac{1}{nh_n}\int_{-\infty}^{\infty} K(\frac{t-s+h_ny}{h_n})K(y)f(s-h_ny)dy + \frac{n-1}{n}\int_{-\infty}^{\infty} k(y)f(t-h_ny)dy\int_{-\infty}^{\infty} K(y)f(s-h_ny)dy$$

$$\to f(t)f(s), n\to\infty, nh_n\to\infty, \frac{n-1}{n}\to 1$$

So $lim_{n\to\infty}E[\int_a^b \hat{f}_n(t)dt]^2 = lim_{n\to\infty}E[\int_a^b \hat{f}_n(t)dt][\int_a^b \hat{f}_n(s)ds]$

$$= lim_{n\to\infty}\int_a^b\int_a^b E[\hat{f}_n(s)\hat{f}_n(t)]dtds$$

$$= lim_{n\to\infty}\int_a^b\int_a^b f(t)f(s)dtds$$

$$= (\int_a^b f(t)dt)^2 = (lim_{n\to\infty} E\int_a^b \hat{f}_n(t)dt)^2$$

So $lim_{n\to\infty} var(\int_a^b \hat{f}_n(t)dt) \to 0$

$$\int_a^b \hat{f}_n(t)dt \xrightarrow{p} \int_a^b f(t)dt$$

Problem 2

$\hat{f}_n(t) = \frac{\hat{F}_n(t+h_n) - \hat{F}_n(t-h_n)}{2h_n}$

$$||\hat{f}_n(t) - f(t)||_\infty \leq ||\hat{f}_n(t) - f_n(t)||_\infty + ||f_n(t) - f(t)||_\infty$$

By definition of $f(t)$, $lim_{n\to\infty}||f_n(t) - f(t)||_\infty = 0$

$$||\hat{f}_n(t) - f(t)||_\infty \leq \frac{1}{2h_n}(||\hat{F}_n(t+h_n) - F(t+h_n)||_\infty + ||\hat{F}_n(t-h_n) - F(t-h_n)||_\infty)$$

So by DKW inequality

$$P(||\hat{f}_n - f||_\infty > \epsilon) \leq 2P(\frac{1}{2h_n}||\hat{F}_n - F||_\infty > \epsilon) = P(||\hat{F}_n - F||_\infty > \epsilon h_n) \leq Ce^{-2nh_n^2\epsilon^2}$$

$$= Ce^{-o(log(n))\epsilon^2} = c[o(\frac{1}{n})]^{\epsilon^2} \to 0$$

Since $\frac{nh_n^2}{log(n)} \to 0 \Rightarrow nh_n^2 = o(log(n))$, By Borel Cantelli lemma:

$$sup_{t\in R}|\hat{f}_n - f(t)| \xrightarrow{a.s.} 0$$