5062 Homework 1 Solution

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1 Problem 1

1.1

Let $T = \sum_{i=1}^n X_i$ and $A = nI_k + \Sigma_0^{-1}$. The product of the density of X and the prior density is

$$C_X exp\{-\frac{||T - n\theta||^2}{2} - \frac{(\theta - \mu_0)^\tau \Sigma_0^{-1}(\theta - \mu_0)}{2}\}$$

$$= D_X exp\{-\frac{[\theta - A^{-1}(\Sigma_0^{-1}\mu_0 + T)]^\tau A[\theta - A^{-1}(\Sigma_0^{-1}\mu_0 + T)]}{2}\},$$

where C_X and D_X are quantities depending on X but not θ. Thus the posterior distribution of θ given X is $N_k(A^{-1}(\Sigma_0^{-1}\mu_0 + T), A^{-1})$. Since the posterior is a normal distribution,

$$E(\theta|X) = (\Sigma_0^{-1}\mu_0 + T)A^{-1}$$

and

$$Var(\theta|X) = A^{-1},$$

where $T = \sum_{i=1}^n X_i$ and $A = nI_k + \Sigma_0^{-1}$.

1.2

Let $T = \sum_{i=1}^n X_i$. The product of the density of X and the prior density is

$$C_X \theta^{T+\alpha-1}(1-\theta)^{nk-T+\beta-1},$$

where C_X does not depend on θ. Thus, the posterior distribution of θ given X is beta distribution with parameter $(T + \alpha, nk - T + \beta)$ and

$$E(\theta|X) = \frac{T+\alpha}{nk+\alpha+\beta}$$

$$Var(\theta|X) = \frac{(T+\alpha)(nk-T+\beta)}{(nk+\alpha+\beta)^2(nk+\alpha+\beta+1)},$$

where $T = \sum_{i=1}^n X_i$.

1.3

Let $X_{(n)}$ be the largest order statistics. The product of the density of X and the prior density is

$$\theta^{-n} I_{(0,\theta)}(X_{(n)}) b a^b \theta^{-(b+1)} I_{(a,\infty)}(\theta) = \theta^{-(n+b+1)} I_{(max X_{(n)}, a, \infty)}(\theta).$$

Thus the posterior distribution of θ given X has the same form as the prior with a replaced by $max\{X_{(n)}, a\}$ and b replaced by $b+n$. After direct calculation,

$$E(\theta|X) = \frac{max\{X_{(n)}, a\}(b+n)}{(b+n-1)}$$

and

$$Var(\theta|X) = \frac{max\{X_{(n)}^2, a^2\}(b+n)}{(b+n-1)^2(b+n-2)}.$$

1.4

Let $T = \sum_{i=1}^n X_i$. The product of the density of X and the prior density is

$$C_X \theta^{-(n+\alpha+1)} exp\{-(T+\gamma^{-1})/\theta\},$$

where C_X does not depend on θ. Thus the posterior distribution of θ given X is the inverse gamma distriution with shape parameter $n+\alpha$ and scale parameter $(T+\gamma^{-1})$. Then

$$E(\theta|X) = \frac{T+\gamma^{-1}}{\alpha+n-1}$$

and

$$Var(\theta|X) = \frac{(T+\gamma^{-1})^2}{(n+\alpha-1)(n+\alpha-2)}.$$

1.5

Let $T = \sum_{i=1}^n X_i$. The product of the density of X and the prior density is

$$C_X exp\{\theta(\frac{1}{b}+n) I_{(-\infty,a)}(\theta) I_{(\theta,\infty)}(X_{(1)})\},$$

where C_X does not depend on X. The posterior of θ given X has the same distribution as the prior with a replaced by $min\{X_{(1)}, a\}$ and b replaced by $\frac{b}{1+nb}$. A direct calculation shows that

$$E(\theta|X) = min(a, X_{(1)}) - \frac{b}{nb+1})$$

$$Var(\theta|X) = \frac{b^2}{(nb+1)^2}.$$

2 Problem 2

2.1

Note that $\bar{X}$ has distribution $N(\theta, \frac{\sigma^2}{n})$. The product of the density of $\bar{X}$ and $\pi(\theta)$ is

$$\frac{\sqrt{n}}{\sqrt{2\pi}\sigma}e^{\frac{-n(x-\theta)^2}{2\sigma^2}}\pi(\theta).$$

Hence,

$$p(x) = \int \frac{\sqrt{n}}{\sqrt{2\pi}\sigma}e^{\frac{-n(x-\theta)^2}{2\sigma^2}}\pi(\theta)dv$$

and

$$p^{'}(x) = \frac{n}{\sigma^2}\int \frac{\sqrt{n}}{\sqrt{2\pi}\sigma}(\theta - x)e^{\frac{-n(x-\theta)^2}{2\sigma^2}}\pi(\theta)dv.$$

Then, the posterior mean is

$$\begin{aligned}\delta(x) &= \frac{1}{p(x)}\int \frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\theta e^{\frac{-n(x-\theta)^2}{2\sigma^2}}\pi(\theta)dv \\ &= x + \frac{1}{p(x)}\int \frac{\sqrt{n}}{\sqrt{2\pi}\sigma}(\theta - x)e^{\frac{-n(x-\theta)^2}{2\sigma^2}}\pi(\theta)dv \\ &= x + \frac{\sigma^2}{n}\frac{p^{'}(x)}{p(x)} \\ &= x + \frac{\sigma^2}{n}\frac{dlog(p(x))}{dx}\end{aligned}$$

2.2

From the result in 2.1,

$$p^{''}(x) = \frac{n^2}{\sigma^4}\int \frac{\sqrt{n}}{\sqrt{2\pi}\sigma}(\theta - x)^2 e^{\frac{-n(x-\theta)^2}{2\sigma^2}}\pi(\theta)dv - \frac{\sigma^2}{n}$$

and therefore,

$$\begin{aligned}Var(\theta|\bar{X} = x) &= E[(\theta - x)^2|\bar{X} = x] - [E(\theta - x|\bar{X} = x)]^2 \\ &= \frac{\sigma^4}{n^2}\frac{p^{''}(x)}{p(x)} + \frac{\sigma^2}{n} - [\frac{\sigma^2}{n}\frac{p^{'}(x)}{p(x)}]^2 \\ &= \frac{\sigma^4}{n^2}\frac{d^2log(p(x))}{dx^2} + \frac{\sigma 2}{n}\end{aligned}$$

2.3

If the prior is $N(\mu_0, \sigma_0^2)$, then the joint distribution of Θ and $\bar{X}$ is normal and, hence, the marginal distribution of $\bar{X}$ is normal. The mean of $\bar{X}$ condition on θ is θ. Hence the marginal mean of $\bar{X}$ is μ_0. The variance of $\bar{X}$ condition on θ is $\frac{\sigma^2}{n}$. Hence the marginal variance of $\bar{X}$ is $\sigma_0^2 + \frac{\sigma^2}{n}$. Thus, p(x) is the density of $N(\mu_0, \sigma_0^2 + \frac{\sigma^2}{n})$ if the prior is $N(\mu_0, \sigma_0^2)$. If the prior is a point mass at μ_1, then

$$p(x) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma}e^{\frac{-n(x-\mu_1)^2}{2\sigma^2}},$$

which is the density of $N(\mu_1, \frac{\sigma^2}{n})$. Therefore, P(x) is the density of the misture distribution $(1-\epsilon)N(\mu_0, \sigma_0^2 + \frac{\sigma^2}{n}) + \epsilon N(\mu_1, \frac{\sigma^2}{n})$. Then

$$p^{'}(x) = \frac{(1-\epsilon)(\mu_0 - x)}{\sigma_0^2 + \frac{\sigma^2}{n}}\phi(\frac{x-\mu_0}{\sqrt{\sigma_0^2 + \frac{\sigma^2}{n}}}) + \frac{\epsilon(\mu_1 - x)}{\frac{\sigma^2}{n}}\phi(\frac{x-\mu_1}{\sqrt{\frac{\sigma^2}{n}}}),$$

where $\phi(x)$ is the standard normal density and $\delta(x)$ can be obtained by the formula in 2.1.

3 Problem 3

3.1

The posterior density $\pi(\mu, \sigma^2|x)$ is proportional to

$$\sigma^{-(n+2)}exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \bar{x})^2 exp\{-\frac{(\mu-\bar{x})^2}{\frac{\sigma^2}{n}}\}I_{(0,\infty)}(\sigma^2),$$

which is proportional to $\pi_1(\mu|\sigma^2, x)\pi_2(\sigma^2|x)$.

3.2

The marginal posterior density of μ is

$$\pi(\mu|x) = \int_0^\infty \pi(\mu, \sigma^2|x)d\sigma^2 \propto \int_0^\infty \sigma^{-(n+2)}exp\{-\frac{n(\bar{x}-\mu)^2}{2\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \bar{x})^2\}d\sigma^2$$

$$\propto [1 + \frac{1}{n-1}(\frac{\mu - \bar{x}}{\tau})^2]^{-\frac{n}{2}}$$

Hence,$\pi(\mu|x)$ is $f(\frac{\mu-\bar{x}}{\tau})$ with f being the density of student t distribution t_{n-1}.

3.3

The generalized Bayes action is

$$\begin{aligned}\delta &= \int \frac{\mu}{\sigma}\pi_1(\mu|\sigma^2, x)\pi_2(\sigma^2|x)d\mu d\sigma^2 \\ &= \bar{x}\int \sigma^{-1}\pi_2(\sigma^2|x)d\sigma^2 \\ &= \frac{\Gamma(n/2)\bar{x}}{\Gamma((n-1)/2)\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2/2}}\end{aligned}$$

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4 Problem 4

4.1

Let weighted function be $W(p) = p^{-1}(1-p)^{-1}$, then $f(x|p) = \prod_{i=1}^n p^{x_i}(1-p)^{1-x_i} = p^{\sum x_i}(1-p)^{n-\sum x_i}$.

$$\delta^*(x) = \frac{\int_0^1 pf(x|p)w(p)dp}{\int_0^1 f(x|p)w(p)dp} = \frac{\int_0^1 p^{\sum x_i}(1-p)^{n-\sum x_i - 1}dp}{\int_0^1 p^{\sum x_i - 1}(1-p)^{n-\sum x_i - 1}dp} = \frac{B(\sum x_i + 1, n - \sum x_i)}{B(\sum x_i, n - \sum x_i)} = \frac{\sum x_i}{n} = \bar{x}$$

4.2

The discrete probability density of X is $(1-p)^{x-1}p$. Hence, for estimating p with loss $(p-a)^2/p$, the Bayes action when X=x is

$$\delta(x) = \frac{\int_0^1 p^{-1}p(1-p)^{x-1}pd\prod}{\int_0^1 p^{-1}(1-p)^{x-1}pd\prod} = 1 - \frac{\int_0^1 (1-p)^x d\prod}{\int_0^1 (1-p)^{x-1}d\prod}.$$

Consider the prior with $Lebesgue$ density $\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}p^{\alpha-1}(1-p)^{\alpha-1}I_{(0,1)(p)}$. The Bayes action is

$$\delta(x) = 1 - \frac{\int_0^1 (1-p)^{x+\alpha-1}p^{\alpha-1}dp}{\int_0^1 (1-p)^{x+\alpha-2}p^{\alpha-1}dp} = 1 - \frac{\frac{\Gamma(x+2\alpha-1}{\Gamma(x+\alpha-1)\Gamma(\alpha)}}{\frac{\Gamma(x+2\alpha)}{\Gamma(x+\alpha)\Gamma(\alpha)}} = 1 - \frac{x+\alpha-1}{x+2\alpha-1}$$

Since $\lim_{\alpha\Rightarrow 0}\delta(x) = \frac{1}{2}I_1(x) = \delta_0(x)$,$\delta_0(x)$ is a limit of Bayes actions. Consider the improper prior density $\frac{d\prod}{dp} = [p^2(1-p)]^{-1}$. Then the posterior risk for action a is

$$\int_0^1 (p-a)^2(1-p)^{x-2}p^{-2}dp.$$

When $x=1$, the above integral diverges to infinity and, therefore, any a is a Bayes action. When $x>1$, the above integral converges if and only if $a=0$. Hence δ_0 is a Bayes action.