7 Lecture 7: Bayesian Prospective of MRIE

Motivating example: location invariant estimator
Suppose the p.d.f. of X satisfies $f(x-\theta), \theta \in \mathbb{R}, x \in \mathbb{R}$, under loss function $L(\theta, a) = (\theta - a)^2$. Our goal is to estimate θ.
Consider $\delta_0(x) = x$, then $\delta(x)$ is invariant if and only if $\delta(x) = \delta_0(x) - u = x - u$.
$\delta^*(x) = x - u^*$ is MRIE if and only if

$$\delta^*(x) = \arg\min_{\delta \in \mathbb{A}} R(\theta, \delta) = \arg\min E_\theta[(x-u)^2] = \arg\min E_0[(x-u)^2].$$

So $\delta^* = E(x|\theta) = E_{\theta=0}(x) = 0$. Let's look at the function to be minimized.

$$\begin{aligned}
E_0[(x-u)^2] &= \int (x'-u)^2 f_0(x')dx' = \int (x' + \delta(x) - x)^2 f_0(x')dx', \quad \text{let } x' = x - \theta, \\
&= \int (x - \theta + \delta(x) - x)^2 f_0(x-\theta)d\theta \\
&= \int (\delta(x) - \theta)^2 f_0(x-\theta)d\theta \\
&= \int (\delta(x) - \theta)^2 \Pi(\theta|x)d\theta \\
&\propto r(\Pi, \delta), \quad \text{Bayes risk w.r.t. } \Pi.
\end{aligned}$$

where Π is the generalized Bayes posterior under uniform prior on $\mathbb{R}$ (improper prior),

$$\Pi(\theta|x) = \frac{\Pi(\theta)f(x|\theta)}{\int \Pi(\theta)f(x|\theta)d\theta} = \frac{f(x|\theta)}{\int f(x|\theta)d\theta} = \frac{f_0(x-\theta)}{\int f(x|\theta)d\theta} f_0(x-\theta).$$

Hence the generalized bayes rule that minimizes the above Bayes risk is

$$\begin{aligned}
\delta^*(x) = E(\theta|x) &= \int \theta \Pi(\theta|x)d\theta \\
&= \int \theta f_0(x-\theta)d\theta, \quad \text{let } x' = x - \theta, \\
&= \int (x - x')f_0(x')dx' \\
&= x\int f_0(x')dx' - \int x' f_0(x')dx' \\
&= x - E(x|\theta = 0) = x
\end{aligned}$$

The generalized Bayes rule w.r.t. uniform prior is sames as MRIE.

Theorem 7.1 (Eaton 1989)**.** *If $\bar{G}$ acts transitive on Θ, the MRIE is given by*

$$\delta^*(x) = \arg\min_{\delta \in \mathbb{A}} \int_{\bar{G}} L(\bar{g}\theta, \delta) f(x|\bar{g}\theta_0) d\mu_r(\bar{g}),$$

where μ_r is the right-invariant measure on $\bar{G}$ and θ_0 is any point in Θ.
(Note: as $\bar{g}$ ranges over $\bar{G}$, $\bar{g}\theta_0$ ranges over entire Θ.)

Definition Right-invariant prior corresponding to $\bar{G}$ on Θ is

$$\forall A \in \sigma(\Theta), \quad \Pi_r(A) = \mu_r\{\bar{g} : \bar{g}\theta_0 \in A\} = \int \mathbf{1}(\bar{g}\theta_0 \in A) d\mu_r(\bar{g}),$$

where θ_0 is any point in Θ, μ_r is the right-invariant measure on $\bar{G}$.

Interpretation: Π_r is the distribution of $\bar{g}\theta_0$, where $\bar{g}$ is randomly selected from $\bar{G}$.

Corollary 7.2. *If $\bar{G}$ is transitive. The MRIE is given by*

$$\delta^*(x) = \arg\min_{\delta \in \mathbb{A}} \int_\Theta L(\theta, \delta) d\Pi_r(\theta|x),$$

where $\Pi_r(\theta|x)$ is the posterior of θ with p.d.f (w.r.t $\Pi_r(\theta)$)

$$\frac{d\Pi_r(\theta|x)}{d\Pi_r(\theta)} = \frac{f(x|\theta)}{\int f(x|\theta) d\Pi_r(\theta)}$$

$$(d\Pi_r(\theta|x) = \frac{f(x|\theta) d\Pi_r(\theta)}{\int f(x|\theta) d\Pi_r(\theta)})$$

Remarks: This corollary shows that the MRIE δ^* can be viewed as a generalized Bayes estimator w.r.t prior Π_r, since

$$\int_\Theta L(\theta, \delta) d\Pi_r(\theta|x) = \frac{1}{C_x} \int_\Theta L(\theta, \delta) f(x|\theta) d\Pi_r(\theta) = \frac{1}{C_x} r(\Pi_r, \delta),$$

where $C_x = \int f(x|\theta) d\Pi_r(\theta)$ is independent of θ.

Finding the Right-invariant Measure μ_r on $\bar{G}$

A measure μ_r is right-invariant iff $\forall \bar{g_0} \in \bar{G}$ and measurable $A \subset \bar{G}$,

$$\mu_r(A\bar{g_0}^{-1}) = \mu_r(A)$$

In other words, if view μ_r as a probablity measure

$$P(\bar{g} \in A) = P(\bar{g} \circ \bar{g_0} \in A)$$

Examples:

1. Additive groups: $\bar{G} = \{\bar{g} : \bar{g}(\theta) = \theta + c, c \in \mathbb{R}\}$.
   $\bar{g_0}(\theta) = \theta + c_0$, then $\bar{g} \circ \bar{g_0} = \theta + (c + c_0)$. μ_r is right-invariant if $\mu_r(A - c_0) = \mu_r(A), \forall c_0 \in \mathbb{R}, \forall A \in \mathcal{B}_\mathbb{R}$
   Lebesgue measure is invariant on $\mathbb{R}$. The induced prior is given by the distribution of $\theta_0 + c$, where $c \sim \mu_r = \mu_L$ (lebesgue measure)
   $\therefore \Pi_r = \mu_r = \mu_L$

2. Multiplicative group: $\bar{G} = \{\bar{g} : \bar{g}(\theta) = c\theta, c > 0\}$.
   $\bar{g_0}(\theta) = c_0\theta$, then $\bar{g} \circ \bar{g_0} = c \cdot c_0\theta$. $\bar{G}$ is isomorphic on $\mathbb{R}^+$. μ_r is right-invariant if

   $$\mu_r(A/c_0) = \mu_r(A), \forall c_0 \in \mathbb{R}^+, \forall A \subset \mathcal{B}_{\mathbb{R}^+}$$

   Lebesgue measure on $\mathbb{R}^+$ is not right- invariant. Consider the measure μ_r with the density $1/c$ w.r.t lebesgue measure μ_L. $\forall B = (b_1, b_2) \in \mathbb{R}^+$, let $c' = c \cdot c_0$ Then

   $$\mu_r(B/c_0) = \int_{b1/c_0}^{b2/c_0} \frac{1}{c} d\mu_L(c) = \int_{b1}^{b2} \frac{c_0}{c'} d\mu_L(\frac{c'}{c_0}) = \int_{b1}^{b2} \frac{1}{c'} d\mu_L(c') = \mu_r(B)$$

   $\therefore \mu_r$ is the right-invariant measure. The induced right-invariant prior $\Pi_r = \mu_r$

7.1 Related Reading

1. Sh P253-255
2. LC Chapter 4