Differential equations: Practice problems for April Exam, 2010

Problem 1

Use the method of elimination to find the $x$-component of the general solution of

\[
\begin{align*}
x'' &= 6x' - 9x + y \\
y'' &= x - 6y' - 9y
\end{align*}
\]

Problem 2

A system of 4 tanks is set up in such a way that solution passes from tank 1 to tank 2 at a rate of 2 gallon per minute, from tank 2 to each of tanks 3 and 4 at a rate of 1 gallon per minute each, from tank 3 to tanks 1 and 4 at a rate of 1 gallon per minute each and from tank 4 to tanks 1 and 3 at a rate of 1 gallon per minute. Each tank contains 1 gallon of solution.

Find a matrix $A$ that describes the system

\[
\frac{d\mathbf{x}}{dt} = A\mathbf{x}
\]

Problem 3

Provide a solution of the system

\[
\begin{bmatrix}
x_1''(t) \\
2x_2''(t) \\
3x_3''(t)
\end{bmatrix} =
\begin{bmatrix}
-2 & 1 & 0 \\
1 & -4 & 3 \\
0 & 3 & -6
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix} + \cos 2t
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

having initial displacement vector

\[
\begin{bmatrix}
x_1(0) \\
x_2(0) \\
x_3(0)
\end{bmatrix} =
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

Problem 4
Two matrices $A$ and $B$ are said to commute if $AB = BA$. Which pairs among the following of matrices commute? Is it true that if $AB = BA$ and $CB = BC$ then $AC = CA$? Is it true that if $AB = BA$ and $CB \neq BC$ then $AC \neq CA$?

\[
A_1 = \begin{bmatrix} 5 & -1 & -1 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 9 & -3 & -3 \\ -3 & 6 & 0 \\ -3 & 0 & 6 \end{bmatrix} \quad A_3 = \begin{bmatrix} 7 & 0 & -4 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{bmatrix}
\]

\[
A_4 = \begin{bmatrix} 11 & -2 & -6 \\ -1 & 4 & 0 \\ -1 & -2 & 6 \end{bmatrix} \quad A_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & 2 \end{bmatrix}
\]

**Problem 5**

Find the general solution of the system

\[
\frac{dx}{dt} = \begin{bmatrix} 3 & 0 & -3 \\ -3 & 6 & 0 \\ 0 & -6 & 3 \end{bmatrix} x
\]

**Problem 6**

A system of 4 masses and 5 springs is as follows: $m_1 = 2$, $m_2 = 4$, $m_3 = 8$, $m_4 = 16$ and $k_1 = 1$, $k_2 = 2$, $k_3 = 4$ and $k_4 = 8$ and $k_5 = 16$. What are the natural frequencies of the system? The determinant of a $4 \times 4$ matrix $A$ whose $(i,j)$-entry is $a_{ij}$ is given by $\det A = a_{11} \det A(1, 1) - a_{12} \det A(1, 2) + a_{13} \det A(1, 3) - a_{14} \det A(1, 4)$ where $A(i,j)$ is the $3 \times 3$ matrix obtained by deleting the $i$th row and $j$th column of $A$.

**Problem 7**

Find the general solution of the following system

\[
\frac{dx}{dt} = \begin{bmatrix} 17 & 1 & 2 \\ -9 & 15 & 0 \\ 4 & -4 & 10 \end{bmatrix} x
\]
Problem 8

Find the exponential of the matrix

\[ tA = t \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \]

Problem 9

For each of the following matrices, find a matrix \( A \) such that the given matrix has the form \( e^{tA} \) or explain why there cannot be such a matrix.

I. \[ \begin{bmatrix} e^{\pi t} & 0 & 0 \\ 0 & e^{e t} & 0 \\ 0 & 0 & e^{\text{one million} t} \end{bmatrix} \]

II. \[ \begin{bmatrix} e^t & \frac{1}{2} t^2 e^t & t e^t \\ 0 & e^t & 0 \\ 0 & t e^t & e^t \end{bmatrix} \]

III. \[ \begin{bmatrix} t e^t \cos t + e^t \sin t & -e^t \sin t \\ e^t \sin t & t e^t \cos t - e^t \sin t \end{bmatrix} \]

IV. \[ \begin{bmatrix} e^{-t} \cos 3t & e^{-t} \sin 3t \\ -e^{-t} \sin 3t & e^{-t} \cos 2t \end{bmatrix} \]

Problem 10

Find the exponential \( e^{tA} \) of the matrix

\[ A = \begin{bmatrix} 4 & 0 & 2 \\ 4 & 2 & 2 \\ -2 & 0 & 0 \end{bmatrix} = 2U^{-1}TU \quad \text{where} \]

\[ 2U^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \]

Problem 11

Solve the initial value problem

\[ \frac{dx}{dt} = \begin{bmatrix} 3 & -9 \\ 5 & -3 \end{bmatrix} x; \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
Problem 12

Use the method of undetermined coefficients to solve

\[ x' = 5x - 6y; \quad y' = -4x + 3y + \sin t, \quad x(0) = -24/82, \ y(0) = -25/82 \]

Problem 13

Use the Laplace transform to solve the initial value problem

\[ x'' + 4x' + 8x = e^{-2t}; \quad x(0) = 0, \ x'(0) = 1. \]

Problem 14

Find the Laplace transform of the function \( f(t) = t - [[t]], \ (t \geq 0) \) where \([ [t] \)] denotes the integer part of \( t \), that is, \([ [t] \] = n \) if \( n \leq t < n + 1 \).

Problem 15

Find the solution of the system

\[
\frac{dx}{dt} = \begin{bmatrix} -4 & 5 & -1 \\ 0 & -6 & 5 \\ 0 & 0 & -4 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\]