

Homework 5

Again, although, I want you to submit only the problems below (+ the extra, if you want), I strongly recommend to you to look at ALL the problems after chapter 4.

Problems for all.

Solve problems (4.)J, K, L, O, R on pp. 37–39 of Bartle's book.

Solutions.

- 4.J. (a) Let $f_n = \frac{1}{n}\chi_{[0,n]}$. For every $\epsilon > 0$ and $x \in \mathbb{R}$ there exists $N > \frac{1}{\epsilon}$ such that $|f_n(x)| < \epsilon$ for every $n > N$. Hence, f_n converges to $f = 0$ uniformly. However,

$$0 = \int f d\lambda \neq \lim \int f_n d\lambda = 1.$$

The MCT does not apply because the sequence is not monotone increasing. Fatou's lemma obviously applies.

- (b) Let $g_n = n\chi_{[\frac{1}{n}, \frac{2}{n}]}$, $g = 0$. Again,

$$0 = \int g d\lambda \neq \lim \int g_n d\lambda = 1.$$

However, this time convergence is not uniform (apply the definition). The MCT still does not apply because the sequence is not monotone increasing and Fatou's lemma does apply.

- 4.K. $f \in M^+$ by Corollary 2.10 and the usual properties of the limit. Moreover, for a given $\epsilon > 0$ let $N \in \mathbb{N}$ be such that $\sup |f(x) - f_n(x)| < \epsilon$ for all $n > N$. Then

$$\left| \int f d\mu - \int f_n d\mu \right| \leq \int \epsilon d\mu = \epsilon \mu(X)$$

implies the desired equality.

- 4.L. See Prof. Wilson's handout.

- 4.O. Apply Fatou's lemma to $f_n + h$.

- 4.R. Let ϕ_n be an increasing sequence of real-valued step functions that converges to f pointwise. Let $\phi_n = \sum_{j=1}^{k_n} \lambda_{j,n} \chi_{E_{j,n}}$ be the canonical representation of ϕ_n . Clearly, $\mu(E_{j,n}) < \infty$ for all j, n because f is integrable. Then

$$N = \bigcup_{n \in \mathbb{N}} \{x \in X : \phi_n(x) > 0\} = \bigcup_{n \in \mathbb{N}} \bigcup_{j=1}^{k_n} E_{j,n}$$

implies that N is σ -finite.

Extra problems.

6* Prove the following easy inequality due to Chebyshev:

For $f \in M^+$ and $E_\alpha = \{x \in X : f(x) \geq \alpha\}$,

$$\mu(E_\alpha) \leq \frac{1}{\alpha} \int f d\mu.$$

Solution. Follows from

$$\frac{1}{\alpha} \int f d\mu \geq \frac{1}{\alpha} \int_{E_\alpha} f d\mu \geq \frac{1}{\alpha} \int_{E_\alpha} \alpha d\mu = \mu(E_\alpha).$$