

# Homework 5

Again, although, I want you to submit only the problems below (+ the extra, if you want), I strongly recommend to you to look at ALL the problems after chapter 4.

## Problems for all.

Solve problems (4.)J, K, L, O, R on pp. 37–39 of Bartle's book.

## Solutions.

- 4.J. (a) Let  $f_n = \frac{1}{n}\chi_{[0,n]}$ . For every  $\epsilon > 0$  and  $x \in \mathbb{R}$  there exists  $N > \frac{1}{\epsilon}$  such that  $|f_n(x)| < \epsilon$  for every  $n > N$ . Hence,  $f_n$  converges to  $f = 0$  uniformly. However,

$$0 = \int f d\lambda \neq \lim \int f_n d\lambda = 1.$$

The MCT does not apply because the sequence is not monotone increasing. Fatou's lemma obviously applies.

- (b) Let  $g_n = n\chi_{[\frac{1}{n}, \frac{2}{n}]}$ ,  $g = 0$ . Again,

$$0 = \int g d\lambda \neq \lim \int g_n d\lambda = 1.$$

However, this time convergence is not uniform (apply the definition). The MCT still does not apply because the sequence is not monotone increasing and Fatou's lemma does apply.

- 4.K.  $f \in M^+$  by Corollary 2.10 and the usual properties of the limit. Moreover, for a given  $\epsilon > 0$  let  $N \in \mathbb{N}$  be such that  $\sup |f(x) - f_n(x)| < \epsilon$  for all  $n > N$ . Then

$$\left| \int f d\mu - \int f_n d\mu \right| \leq \int \epsilon d\mu = \epsilon \mu(X)$$

implies the desired equality.

- 4.L. See Prof. Wilson's handout.

- 4.O. Apply Fatou's lemma to  $f_n + h$ .

- 4.R. Let  $\phi_n$  be an increasing sequence of real-valued step functions that converges to  $f$  pointwise. Let  $\phi_n = \sum_{j=1}^{k_n} \lambda_{j,n} \chi_{E_{j,n}}$  be the canonical representation of  $\phi_n$ . Clearly,  $\mu(E_{j,n}) < \infty$  for all  $j, n$  because  $f$  is integrable. Then

$$N = \bigcup_{n \in \mathbb{N}} \{x \in X : \phi_n(x) > 0\} = \bigcup_{n \in \mathbb{N}} \bigcup_{j=1}^{k_n} E_{j,n}$$

implies that  $N$  is  $\sigma$ -finite.

## Extra problems.

6\* Prove the following easy inequality due to Chebyshev:

For  $f \in M^+$  and  $E_\alpha = \{x \in X : f(x) \geq \alpha\}$ ,

$$\mu(E_\alpha) \leq \frac{1}{\alpha} \int f d\mu.$$

*Solution.* Follows from

$$\frac{1}{\alpha} \int f d\mu \geq \frac{1}{\alpha} \int_{E_\alpha} f d\mu \geq \frac{1}{\alpha} \int_{E_\alpha} \alpha d\mu = \mu(E_\alpha).$$